Sample Question Paper (Solved)–2025

(Issued by Central Board of Secondary Education, New Delhi)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions :

Read the following instructions very carefully and strictly follow them :

(*i*) *This question paper contains* ***38*** *questions.* ***All*** *questions are* ***compulsory.***
(*ii*) *This question paper is divided into* ***five*** *Sections–****A, B, C, D*** *and* ***E.***
(*iii*) *In* ***Section-A,*** *Questions no.* **1** *to* **18** *are multiple choice questions (MCQs) and questions number* **19** *and* ***20*** *are Assertion-Reason based questions of* ***1*** *mark each.*
(*iv*) *In* ***Section-B,*** *Questions no.* ***21*** *to* ***25*** *are very short answer (VSA) type questions, carrying* ***2*** *marks each.*
(*v*) *In* ***Section-C,*** *Questions no.* ***26*** *to* ***31*** *are short answer (SA) type questions, carrying* ***3*** *marks each.*
(*vi*) *In* ***Section–D,*** *Questions no.* ***32*** *to* ***35*** *are long answer (LA) type questions carrying* ***5*** *marks each.*
(*vii*) *In* ***Section-E,*** *Questions no.* ***36*** *to* ***38*** *are* ***Case Study-based*** *questions carrying* ***4*** *marks each.*
(*viii*) *There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.*
(*ix*) *Use of calculators is* ***not*** *allowed.*

SECTION—A

[1 × 20 = 20]

(*This section comprises of multiple choice questions (MCQs) of 1 mark each*)

Select the correct option (Question 1-Question 18) :

Q. 1. If for a square matrix A,

$$\mathbf{A(adj\ A)} = \begin{bmatrix} 2025 & 0 & 0 \\ 0 & 2025 & 0 \\ 0 & 0 & 2025 \end{bmatrix}, \textbf{ then the}$$

value of | A | + | adj A | is equal to :

(***a***) **1** (***b***) **2025 + 1**

(***c***) **$(2025)^2 + 45$** (***d***) **$2025 + (2025)^2$.**

Ans. (*d*)

Explanation : When A is a square matrix of order $n \times n$, then A.(adj A) = $|A| I_n$, where I_n is the identity matrix of order $n \times n$

$$\therefore \quad A.(\text{adj } A) = \begin{bmatrix} 2025 & 0 & 0 \\ 0 & 2025 & 0 \\ 0 & 0 & 2025 \end{bmatrix}$$

$$= 2025\, I_3$$

$$\therefore \quad |A| = 2025 \text{ and } |\text{adj } A| = |A|^{3-1}$$

$$= (2025)^2.$$

Hence, $|A| + |\text{adj } A| = 2025 + (2025)^2$.

Q. 2. Assume X, Y,Z, W and P are matrices of order 2 × n, 3 × k, 2 × p, n × 3 and p × k, respectively. Then the restriction on n, k and p so that PY + WY will be defined as :

(*a*) $k = 3, p = n$ (*b*) k is arbitrary, $p = 2$

(*c*) p is arbitrary, $k = 3$ (*d*) $k = 2, p = 3$.

Ans. (*a*)

Explanation : P Y W Y

$p \times k$ $3 \times k$ $n \times 3$ $3 \times k$

∴ PY exists if $k = 3$ order of WY = $n \times k$

∴ Order of PY = $p \times k$.

For existence of PY + WY, order of PY = order of WY

∴ $p = n$.

Hence, $k = 3$ and $p = n$.

Q. 3. The interval in which the function *f* defined by f (x) = e^x is strictly increasing is :

(*a*) $[1, \infty)$ (*b*) $(-\infty, 0)$

(*c*) $(-\infty, \infty)$ (*d*) $(0, \infty)$.

Ans. (*c*)

Explanation : $y = e^x \Rightarrow \dfrac{dy}{dx} = e^x$.

For strictly increasing, $\dfrac{dy}{dx} > 0$ *i.e.* $e^x > 0$.

Hence, the function is strictly increasing in $(-\infty, \infty)$.

Q. 4. If A and B are non-singular matrices of same order with det (A) = 5, then det $(B^{-1}AB)^2$ is equal to :

(*a*) 5 (*b*) 5^2

(*c*) 5^4 (*d*) 5^5.

Ans. (*b*)

Explanation : | A | = 5

Now $|B^{-1}AB|^2 = (|B^{-1}||A||B|)^2 = |A|^2 = 5^2$.

Q. 5. The value of 'n', such that the differential equation :

$$x^n \frac{dy}{dx} = y(\log y - \log x + 1); \text{ (where } x, y \in R^+\text{) is}$$

homogeneous is :

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3.

Ans. (*b*)

Explanation : $\frac{dy}{dx} = f(x)$ is homogeneous if $f(x)$ is a homogeneous function of degree 0.

$$\therefore \quad x^n \frac{dy}{dx} = y\left(\log\frac{y}{x} + \log_e e\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x^n}\left(\log_e e\left(\frac{y}{x}\right)\right) = f(x, y).$$

$\therefore$ $f(x, y)$ is homogeneous of degree 0 if $n = 1$.

Q. 6. If the points (x_1, y_1), (x_2, y_2) and $(x_1 + x_2, y_1 + y_2)$ are collinear, then x_1y_2 is equal to :

(*a*) x_2y_1 (*b*) x_1y_1

(*c*) x_2y_2 (*d*) x_1x_2.

Ans. (*a*)

Explanation : (x_1, y_1); (x_2, y_2) and $(x_1 + x_2, y_1 + y_2)$ are collinear

$$\text{if} \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_1 + x_2 & y_1 + y_2 & 1 \end{vmatrix} = 0$$

if $1.(x_2y_1 + x_2y_2 - x_1y_1 - x_2y_2) - 1.(x_1y_1 + x_1y_2 - x_1y_1 - x_2y_1) + 1.(x_1y_2 - x_2y_1) = 0$

if $x_1y_2 = x_2y_1$.

Q. 7. If A = $\begin{bmatrix} 0 & 1 & c \\ -1 & a & -b \\ 2 & 3 & 0 \end{bmatrix}$ is a skew-symmetric matrix, then a + b + c =

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4.

Ans. (*a*)

Explanation : A is skew-symmetric $\Rightarrow A^t = -A$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 2 \\ 1 & a & 3 \\ c & -b & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & c \\ -1 & a & -b \\ 2 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -c \\ 1 & -a & b \\ -2 & -3 & 0 \end{bmatrix}.$$

Comparing $a = -a$, $c = -2$, $-b = -3$

$\Rightarrow$ $a = 0$, $b = 3$ and $c = -2$.

$\therefore$ $a + b + c = 0 + 3 - 2 = 1$.

Q. 8. For any two events A and B, if :

$$P(\overline{A}) = \frac{1}{2}, P(\overline{B}) = \frac{2}{3} \text{ and } P(A \cap B) = \frac{1}{4}, \text{ then}$$

$P(\overline{A}/\overline{B})$ equals :

(*a*) $\frac{3}{8}$ (*b*) $\frac{8}{9}$

(*c*) $\frac{5}{8}$ (*d*) $\frac{1}{4}$.

Ans. (*c*)

Explanation : $P(A) = 1 - \frac{1}{2} = \frac{1}{2}$, $P(B) = 1 - \frac{2}{3} = \frac{1}{3}$.

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}.$$

$$\therefore \quad P(\overline{A}/\overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{P(\overline{A \cup B})}{P(\overline{B})}$$

$$= \frac{1 - \frac{7}{12}}{2/3} = \frac{5/12}{2/3} = \frac{15}{24} = \frac{5}{8}.$$

Q. 9. The value of 'α' if the angle between :

$$\vec{p} = 2\alpha^2\hat{i} - 3\alpha\hat{j} + \hat{k} \text{ and } \vec{q} = \hat{i} + \hat{j} + \alpha\hat{k} \text{ is obtuse is :}$$

(*a*) **R** – [0, 1] (*b*) (0, 1)

(*c*) [0, ∞) (*d*) [1, ∞)

Ans. (*b*)

Explanation : For obtuse angle, $\cos\theta < 0 \Rightarrow \vec{p}.\vec{q} < 0$

$\Rightarrow$ $(2\alpha^2)(1) + (-3\alpha)(1) + (1)(\alpha) < 0$

$\Rightarrow \quad 2\alpha^2 - 3\alpha + \alpha < 0 \Rightarrow 2\alpha^2 - 2\alpha < 0$

$\Rightarrow \quad \alpha^2 - \alpha < 0 \Rightarrow \alpha(\alpha - 1) < 0$

$\Rightarrow \quad \alpha \in (0, 1).$

Q. 10. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{a} + \vec{b}| = 5$, then $|\vec{a} - \vec{b}| =$

(*a*) 3 (*b*) 4

(*c*) 5 (*d*) 8.

Ans. (*c*)

Explanation : $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$

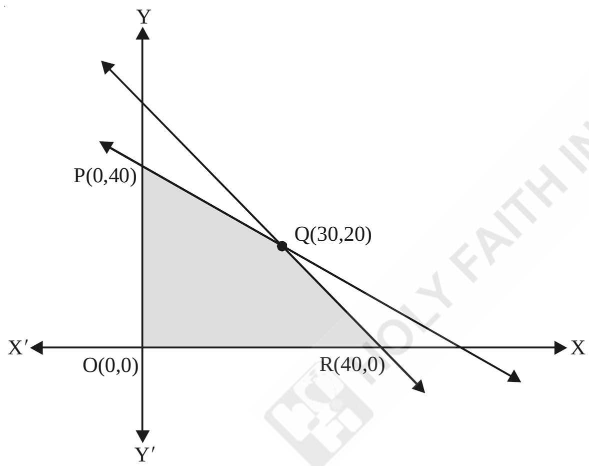
$\Rightarrow \quad 5^2 + |\vec{a} - \vec{b}|^2 = 2(3^2 + 4^2)$

$\Rightarrow \quad |\vec{a} - \vec{b}|^2 = 2(9 + 16) - 25$

$= 50 - 25 = 25.$

Hence, $|\vec{a} - \vec{b}| = 5.$

Q. 11. For the linear programming problem (LPP), the objective function is Z = 4x + 3y and the feasible region determined by a set of constraints is shown in the graph.



Which of the following statement is true ?

(a) Maximum value of Z is at R (40, 0)

(b) Maximum value of Z is at Q (30, 20)

(c) Value of Z at F (40, 0) is less than the value at P (0, 40)

(d) The value of Z at Q (30, 20) is less then the value at R (40, 0).

Ans. (*b*)

Explanation :

Corner Point	Corresponding value of Z = 4x + 3y
O : (0, 0)	0
R : (40, 0)	160
Q : (30, 20)	**180 (Maximum)**
P : (0, 40)	120

Maximum value of Z is at Q (30, 20).

Q. 12. $\int \dfrac{dx}{x^3(1+x^4)^{1/2}}$ equals :

(*a*) $-\dfrac{1}{2x^2}\sqrt{1+x^4} + c$

(*b*) $\dfrac{1}{2x}\sqrt{1+x^4} + c$

(*c*) $-\dfrac{1}{4x}\sqrt{1+x^4} + c$

(*d*) $\dfrac{1}{4x^2}\sqrt{1+x^4} + c.$

Ans. (*a*)

Explanation : $I = \int \dfrac{dx}{x^3(1+x^4)^{1/2}}$

$= \int \dfrac{dx}{x^5\left(1+\dfrac{1}{x^4}\right)^{1/2}}$

Put $1 + \dfrac{1}{x^4} = t$ so that $-4x^{-5}\,dx = dt$

$\Rightarrow \quad \dfrac{dx}{x^5} = -\dfrac{1}{4}dt.$

$\therefore \quad I = \int \dfrac{1}{t^{1/2}}\left(-\dfrac{1}{4}\right)dt$

$= -\dfrac{1}{4}\int t^{-1/2}\,dt$

$= -\dfrac{1}{4}\dfrac{t^{1/2}}{1/2} + c = -\dfrac{1}{2}\sqrt{t} + c$

$= -\dfrac{1}{2}\sqrt{1+\dfrac{1}{x^4}} + c.$

Q. 13. $\displaystyle\int_0^{2\pi} \text{cosec}^7 x\,dx =$

(*a*) 0 (*b*) 1

(*c*) 4 (*d*) 2π.

Ans. (*a*)

Explanation : Here $f(x) = \text{cosec}^7 x.$

$\therefore \quad f(2\pi - x) = \text{cosec}^7(2\pi - x)$

$= -\text{cosec}^7 x = -f(x).$

$$\therefore \int_0^{2\pi} \text{cosec}^7 x \, dx = 0.$$

$$\left[\because \int_0^{2a} f(x)\,dx = 0, \text{if } f(2a - x) = -f(x)\right]$$

Q. 14. What is the general solution of the differential equation $e^{y'} = x$?

(*a*) $y = x \log x + c$

(*b*) $y = x \log x - x + c$

(*c*) $y = x \log x + x + c$

(*d*) $y = x + c.$

Ans. (*b*)

Explanation : We have : $e^{y'} = x$

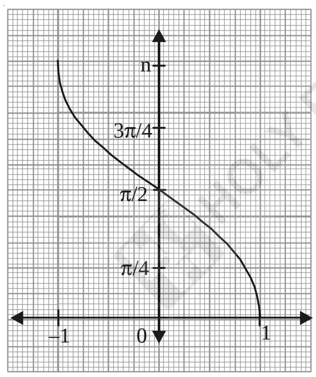
$$\Rightarrow \quad y' = \log x \Rightarrow \frac{dy}{dx} = \log x$$

$$\Rightarrow \quad dy = \log x \, dx.$$

$$\therefore \quad \int dy = \int \log x \, dx$$

$$\Rightarrow \quad y = x \log x - x + c.$$

Q. 15. The graph drawn below depicts.



(*a*) $y = \sin^{-1} x$ (*b*) $y = \cos^{-1} x$

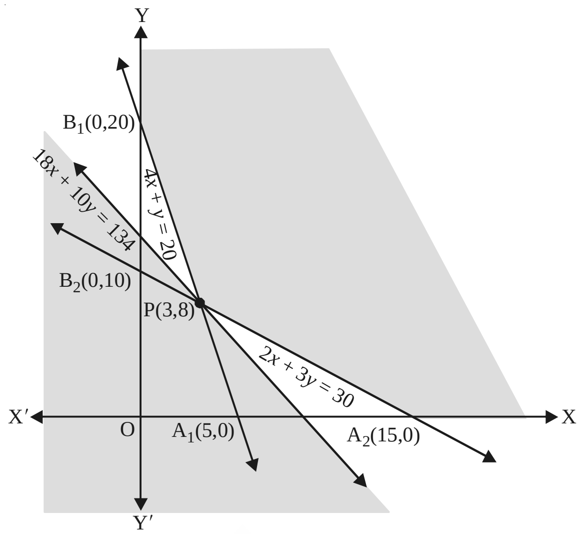
(*c*) $y = \text{cosec}^{-1} x$ (*d*) $y = \cot^{-1} x.$

Ans. (*b*)

Explanation : The graph represents $y = \cos^{-1} x$ with domain $[-1, 1]$ and range $[0, \pi]$

Q. 16. A linear programming problem (LPP) along with the graph of its constraints is shown below :

The corresponding objective function is : Z = 18x + 10y, which has to be minimized. The smallest value of the objective function Z is 134 and is obtained at the corner point (3, 8).



(Note : *The figure is not to scale.)*

The optimal solution of the above linear programming problem ....

(*a*) does not exist as the feasible region is unbounded.

(*b*) does not exist as the inequality $18x + 10y < 134$ does not have any point in common with the feasible region.

(*c*) exists as the inequality $18x + 10y > 134$ has infinitely many points in common with the feasible region.

(*d*) exists as the inequality $18x + 10y < 134$ does not have any point in common with the feasible region.

Ans. (*d*)

Explanation : Since $Z = 18x + 10y < 134$ has no point in common with the feasible region,

$\therefore$ Minimum value of $Z = 18x + 10y$ is 134 at P (3, 8).

Q. 17. The function f : R → Z defined by *f* (*x*) = [*x*]; where[.] denotes the greatest integer function is :

(*a*) Continuous at $x = 2.5$ but not differentiable at $x = 2.5$

(*b*) Not Continuous at $x = 2.5$ but differentiable at $x = 2.5$

(*c*) Not Continuous at $x = 2.5$ and not differentiable at $x = 2.5$

(*d*) Continuous as well as differentiable at $x = 2.5$.

Ans. (*d*)

Explanation : The graph of f : **R** → **R** defined by :

$f(x) = [x]$ is a st. line $\forall\, x \in (2.5 - h, 2.5 + h)$, where h is infinitely small quality.

Hence, the function is continuous and differentiable at $x = 2.5$

Q. 18. A student observes an open air Honeybee nest on the branch of a tree, whose plane figure is parabolic shape given by $x^2 = 4y$. Then the area (in eq. units) of the region bounded by parabola $x^2 = 4y$ and the line $y = 4$ is :

(*a*) $\frac{32}{3}$ (*b*) $\frac{64}{3}$

(*c*) $\frac{128}{3}$ (*d*) $\frac{256}{3}$.

Ans. (*b*)

Explanation : Reqd. area = $\left| 2\int_0^4 2\sqrt{y}\,dy \right|$

$$= 4\left[\frac{y^{3/2}}{3/2}\right]_0^4$$

$$= \frac{8}{3}[4^{3/2} - 0]$$

$$= \frac{8}{3}(8) = \frac{64}{3} \text{ sq. units}$$

ASSERTION-REASON BASED QUESTIONS

(Question numbers 19 and 20 are Assertion-Reason based questions carrying 1 mark each. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the options (A), (B), (C) and (D) as given below.)

(*a*) Both (A) and (R) are true and (R) is the correct explanation of (A).

(*b*) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(*c*) (A) is true but (R) is false

(*d*) (A) is false but (R) is true.

Q. 19. Assertion (A) : Consider the function defined as $f(x) = |x| + |x-1|, x \in \mathbf{R}$. Then $f(x)$ is not differentiable at $x = 0$ and $x = 1$.

Reason (R) : Supose f be defined and continuous on (a, b) and $c \in \mathbf{(a, b)}$, then $f(x)$ is not differentiable at $x = c$ if $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \neq \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$

Ans. (*a*)

Solution. ***When x = 0,*** the function, which involves $|x|$, is not differentiable. [$\because$ *LHD* $\neq$ *RHD*]

When x = 1, the function, which involves $|x - 1|$ is not differentiable. [$\therefore$ *LHD* $\neq$ *RHD*]

$\therefore f(x)$ is not differentiable at $x = 0, 1$.

Thus (A) is true.

Reason (R) is also true.

Hence, (A) and (R) are both true and (R) is the correct explanation of (A).

Q. 20. Assertion (A) : The function $f : R - \left\{(2n+1)\frac{n}{2}; n \in \mathbf{Z}\right\} \to (-\infty, -1] \cup [1, \infty)$ defined by $f(x) = \sec x$ is not one-one function in its domain.

Reason (R) : The line $y = \mathbf{2}$ meets the graph of the function at more than one point.

Ans. (*a*)

Solution. The function $f(x) = \sec x$ is not one-one in its domain because $\sec(x) = \frac{1}{\cos x}$ is periodic with period 2π.

$\therefore$ The function repeats its value after every interval of 2π

$\Rightarrow$ it is not one-one over any interval larger then 2π

[$\because$ *Different inputs give same output*]

Thus (A) is true.

Reason (R) is also true.

[$\because$ *y = 2 meets the graph of the function more than once*]

Hence (A) and (R) are both true and (R) is the correct explanation of (A).

SECTION—B

[2 × 5 = 10]

(This section comprises of 5 very short answer (VSA) type questions of 2 marks each.)

Q. 21. If $\cot^{-1}(3x+5) > \frac{\pi}{4}$, then find the range of the values of x. **Marking Sceme**

Solution. We have : $\cot^{-1}(3x+5) > \frac{\pi}{4}$

$\Rightarrow \cot^{-1}(3x+5) > \cot^{-1} 1$ ½

$\Rightarrow 3x+5 < 1$ [$\because$ *$\cot^{-1} x$ is strictly decreasing in its domain*] ½

$\Rightarrow 3x < -4 \Rightarrow x < -\frac{4}{3}$.

Hence, range is $\left(-\infty, -\frac{4}{3}\right)$. **1**

Q. 22. The cost (in rupees) of producing x items in factory each day is given by :

$C(x) = 0.00013x^3 + 0.002x^2 + 5x + 2200$.

Find the marginal cost when 150 items are produced.

Solution. We have :

$C(x) = 0.00013x^3 + 0.002x^2 + 5x + 2200$.

$\therefore C'(x) = 0.00039x^2 + 0.004x + 5$. **1**

Hence, marginal cost, C′ (150)

= 0.000039 (150)^2 + 0.004 (100) + 5 = ₹ 14.325. **1**

Q. 23. (a) Find the derivative of $\tan^{-1} x$ with respect to $\log_e x$, (where $x \in (1, \infty)$].

Solution. Let $y = \tan^{-1} x$ and $u = \log_e x$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x^2}$$ ½

$$\text{and} \quad \frac{du}{dx} = \frac{1}{x}.$$ ½

$$\therefore \quad \frac{dy}{du} = \frac{dy/dx}{du/dx}$$ ½

$$= \frac{\frac{1}{1+x^2}}{\frac{1}{x}} = \frac{x}{1+x^2}.$$ ½

Or

(b) Differentiate the following function with respect to x :

$(\cos x)^x$; (where $x \in \left(0, \frac{\pi}{2}\right)$).

Solution. Let $y = (\cos x)^x$ ....(1)

Taking logs, $\log y = \log_e (\cos x)^x$ ½

$\Rightarrow \log y = x \log_e (\cos x)$.

Diff. w.r.t x, $\frac{1}{y}\cdot\frac{dy}{dx} = x.\frac{1}{\cos x}(-\sin x) + \log_e(\cos x).1$ ½

$$\Rightarrow \frac{dy}{dx} = y[-x\tan x + \log_e \cos x]$$

Hence, $\frac{dy}{dx} = (\cos x)^x [-x \tan x + \log_e x]$. **1**

Q. 24. (a) If vectors $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{b} + \lambda\vec{c}$ is perpendicular to $\vec{a}$, then find the value of 'λ'.

Solution. We have :

$\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$.

$\therefore \quad \vec{b} + \lambda\vec{c} = (-\hat{i} + 2\hat{j} + \hat{k}) + \lambda(3\hat{i} + \hat{j}) = (-1 + 3\lambda)\hat{i} + (2 + \lambda)\hat{j} + \hat{k}$. ½

Since $\vec{b} + \lambda\vec{c}$ is perpendicular to $\vec{a}$,

$\therefore \quad (\vec{b} + \lambda\vec{c}).\vec{a} = 0$

$\Rightarrow \quad (-1 + 3\lambda)(2) + (2 + \lambda)(2) + (1)(3) = 0$ 1

$\Rightarrow \quad -2 + 6\lambda + 4 + 2\lambda + 3 = 0$

$\Rightarrow \quad 8\lambda = -5$.

Hence, $\lambda = -\frac{5}{8}$. ½

Or

(b) A person standing at O (0, 0, 0) is watching an aeroplane, which is at the co-ordinate point A (4, 0, 3). At the same time he be saw a bird at the coordinate point B (0, 0, 1). Find the angles which $\overrightarrow{BA}$ makes with the x, y and z axes.

Solution. We have : O (0, 0, 0), A (4, 0, 3) and B (0, 0, 1).

$\therefore \quad \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$ ½

$= (4\hat{i} + 3\hat{k}) - \hat{k} = 4\hat{i} + 2\hat{k}$.

$\therefore \quad \hat{BA} = \frac{4}{2\sqrt{5}}\hat{i} + \frac{2}{2\sqrt{5}}\hat{k} = \frac{2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{k}$. ½

Hence, the reqd. angles are $\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$, $\frac{\pi}{2}$ and $\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$. 1

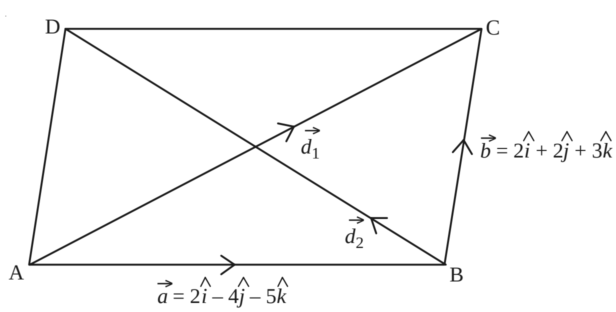
Q. 25. The two co-initial adjacent sides of a parallelogram are :

$2\hat{i} - 4\hat{j} - 5\hat{k}$ and $2\hat{i} + 2\hat{j} + 3\hat{k}$.

Find its diagonals and use them to find the area of the parallelogram.

Solution. Let $\vec{a} = \overrightarrow{AB} = 2\hat{i} - 4\hat{j} - 5\hat{k}$ and $\vec{b} = \overrightarrow{BC} = 2\hat{i} + 2\hat{j} + 3\hat{k}$.

$\therefore \quad \vec{d_1} = \vec{a} + \vec{b} = 4\hat{i} - 2\hat{j} - 2\hat{k}$



and $\vec{d_2} = \vec{a} - \vec{b} = -6\hat{j} - 8\hat{k}.$ ½

$\therefore$ Area of parallelogram $= \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$

$$= \frac{1}{2}\left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & -6 & -8 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\left| 6(-2\hat{i} - 4\hat{k}) - 8(-2\hat{i} - 4\hat{j}) \right| = \frac{1}{2}\left| -12\hat{i} - 24\hat{k} + 16\hat{i} + 32\hat{j} \right|$$

$$= \frac{1}{2}\left| 4\hat{i} + 32\hat{j} - 24\hat{k} \right| = 2\left| \hat{i} + 8\hat{j} - 6\hat{k} \right|$$ 1

$$= 2\sqrt{1 + 64 + 36} = 2\sqrt{101} \text{ sq. units.}$$ ½

SECTION—C

[3 × 6 = 18]

(This section comprises of 6 short answer (SA) type questions of 3 marks each)

Q. 26. A kite is flying at a height of 3 metres and 5 metres of string is out. If the kite is moving away horizontally at the rate of 200 cm/s, find the rate at which the string is being released.

Solution. Let x be the horizontal distance of the kite from the man who flies the kite.

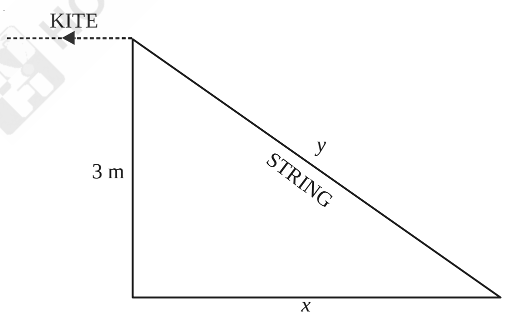
Let y be the length of string. ½

By *Pythagoras' Theorem,* $x^2 + 3^2 = y^2$ ....(1) ½

When $y = 5$, then $x^2 + 9 = 25 \Rightarrow x^2 = 16 \Rightarrow x = 4$. 1

Diff. (1) w.r.t. x, $2x\frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$

$\Rightarrow$ $2(4)(200) = 2(5)\frac{dy}{dt}$



$\Rightarrow$ $\frac{dy}{dt} = 160$ cm/s.

Hence, the rate at which the string is released is 160 cm/s.

Q. 27. According to a psychologist, the ability of a person to understand spatial concept is given by :

$$\mathbf{A = \frac{1}{3}\sqrt{t},} \text{ where } t \text{ is the age in years } t \in [5, 18].$$

Show that the rate of increase of the ability to understand spatial concept decreases with age in between 5 and 18.

Solution. We have : $A = \frac{1}{3}\sqrt{t}$. 1

$$\therefore \quad \frac{dA}{dt} = \frac{1}{6}t^{-1/2} = \frac{1}{6\sqrt{t}} \;\forall\; t \in (5, 18)$$

Thus, $\frac{dA}{dt} = \frac{1}{6\sqrt{t}}$. ½

$$\therefore \quad \frac{d^2A}{dt^2} = \frac{1}{6}\left(-\frac{1}{2}\right)t^{-3/2} = -\frac{1}{12}\frac{1}{t\sqrt{t}}.$$

Clearly $\frac{d^2A}{dt^2} < 0 \;\forall\; t \in (5, 18)$.

Hence, the rate of change of ability to understand spatial concept decreases with age. ½

Q. 28. (a) An ant is moving along the vector $\vec{l_1} = \hat{i} - 2\hat{j} + 3\hat{k}$. Few sugar crystals are kept along the vector $\vec{l_2} = 3\hat{i} - 2\hat{j} + \hat{k}$, which is inclined at an angle θ with the vector $\vec{l_1}$. Then find the angle θ. Also, find the scalar projection of $\vec{l_1}$ on $\vec{l_2}$.

Solution. We have :

$\vec{l_1} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{l_2} = 3\hat{i} - 2\hat{j} + \hat{k}$.

(*i*) $\cos\theta = \frac{\vec{l_1}.\vec{l_2}}{|\vec{l_1}||\vec{l_2}|}$

$$= \frac{(\hat{i} - 2\hat{j} + 3\hat{k}).(3\hat{i} - 2\hat{j} + \hat{k})}{|\hat{i} - 2\hat{j} + 3\hat{k}||3\hat{i} - 2\hat{j} + \hat{k}|}$$ 1

$$= \frac{(1)(3) + (-2)(-2) + (3)(1)}{\sqrt{1+4+9}\sqrt{9+4+1}}$$

$$= \frac{3+4+3}{\sqrt{14}\sqrt{14}} = \frac{10}{14} = \frac{5}{7}.$$

Hence, $\theta = \cos^{-1}\frac{5}{7}$. ½

(*ii*) Scalar projection of $\vec{l_1}$ on $\vec{l_2} = \frac{\vec{l_1}.\vec{l_2}}{|\vec{l_2}|}$

$$= \frac{(\hat{i} - 2\hat{j} + 3\hat{k}).(3\hat{i} - 2\hat{j} + \hat{k})}{|3\hat{i} - 2\hat{j} + \hat{k}|}$$ 1

$$= \frac{(1)(3) + (-2)(-2) + (3)(1)}{\sqrt{9+4+1}} = \frac{3+4+3}{\sqrt{14}} = \frac{10}{\sqrt{14}}.$$ ½

(b) Find the vector and the cartesian equation of the line that passes through (– 1, 2, 7) and is perpendicular to the lines :

$$\vec{r} = 2\hat{i} + \hat{j} - 3\hat{k} + \lambda(\hat{i} + 2\hat{j} + 5\hat{k}) \text{ and } \vec{r} = 3\hat{i} + 3\hat{j} - 7\hat{k} + \mu(3\hat{i} - 2\hat{j} + 5\hat{k}).$$

Solution. The given lines are :

$$\vec{r} = 2\hat{i} + \hat{j} - 3\hat{k} + \lambda(\hat{i} + 2\hat{j} + 5\hat{k}) \quad ...(1)$$

and $\vec{r} = 3\hat{i} + 3\hat{j} - 7\hat{k} + \mu(3\hat{i} - 2\hat{j} + 5\hat{k})$ ...(2)

The line perpendicular to (1) and (2) has a vector parallel to :

$$\vec{b} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 5 \\ 3 & -2 & 5 \end{vmatrix}$$

$$= \hat{i}(10 + 10) - \hat{j}(5 - 15) + \hat{k}(-2 - 6) = 20\hat{i} + 10\hat{j} - 8\hat{k}.$$ 1

Hence, the vector equation of the line is :

$$\vec{r} = -\hat{i} + 2\hat{j} + 7\hat{k} + \lambda(10\hat{i} + 5\hat{j} - 4\hat{k})$$ 1

and the Cartesian equation of the line is : $\dfrac{x+1}{10} = \dfrac{y-2}{5} = \dfrac{z-7}{-4}$. 1

Q. 29. (a) Evaluate :

$$\int \left(\frac{1}{\log x} - \frac{1}{(\log x)^2}\right) dx; \text{ (where } x > 1).$$

Solution. $I = \int \left(\dfrac{1}{\log_e x} - \dfrac{1}{(\log_e x)^2}\right) dx$

$$= \int \frac{dx}{\log_e x} - \int \frac{1}{(\log_e x)^2} dx$$

$$= \frac{1}{\log_e x} \int 1\,dx - \int \left\{\frac{d}{dx}\left(\frac{1}{\log_e x}\right) \int 1\,dx\right\}.dx - \int \frac{1}{(\log_e x)^2\,dx}$$ [*Integating by parts*] 1

$$= \frac{x}{\log_e x} + \int \frac{1}{(\log_e x)^2}.\frac{1}{x}.x\,dx - \int \frac{1}{(\log_e x)^2} dx$$ 1

$$= \frac{x}{\log_e x} + c.$$ 1

(b) Evaluate : $\int_0^1 x\,(1-x)^n\,dx$; (where n ∈ N).

Solution. $I = \int_0^1 x\,(1-x)^n\,dx$

$$= \int_0^1 (1-x)\,\{1-(1-x)\}^n\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$ 1

$$= \int_0^1 x^n\,(1-x)\,dx = \int_0^1 x^n\,dx - \int_0^1 x^{n+1}\,dx$$ ½

$$= \left[\frac{x^{n+1}}{x+1}\right]_0^1 - \left[\frac{x^{n+2}}{n+2}\right]_0^1 = \left(\frac{1}{n+1} - 0\right) - \left(\frac{1}{n+2} - 0\right)$$ ½

$$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{(n+2)-(n+1)}{(n+1)\,(n+2)} = \frac{1}{(n+1)\,(n+2)}\,.$$ 1

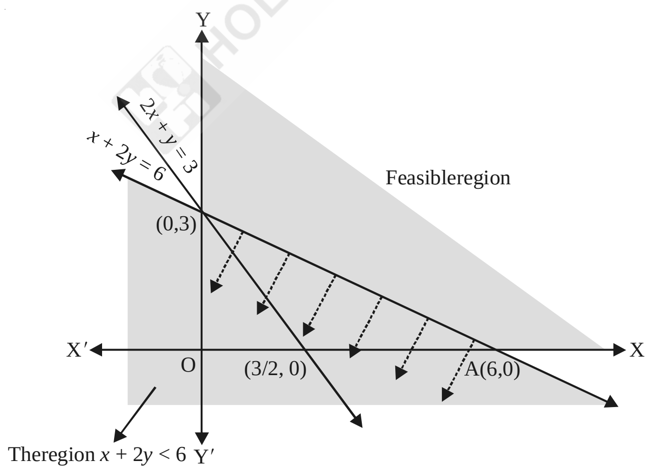
Q. 30. Consider the following linear programming problem :
Minimize Z = x + 2y
Subject to 2x + y ≥ 3, x + 2y ≥ 6, x, y ≥ 0.
Show graphically that the minimum of Z occurs at more than two points :

Solution. The system of constraints is :

$2x + y \geq 3$ ....(1)

$x + 2y \geq 6$ ....(2)

and $x, y \geq 0$ ....(3)



1

The shaded region in the above figure is the feasible region determined by the system of constraints (1) – (3).

It is observed that the feasible region is unbounded.

Thus we use ***Corner Point Method*** to determine the minimum value of Z, where Z = $x + 2y$ ...(4)

We evaluate Z at each corner point.

Corner Point	Corresponding value of z
C : (6, 0)	6
B : (0, 3)	6

1

It is observed that the region $x + 2y < 6$ has no point in in common with unbounded feasible region.
Hence, the minimum value of Z = 6. **1**
It is seen that the value of Z at A and B is same.
When we take any other point; say (2, 2) on the line $x + 2y = 6$, then Z = 6.
Hence, the minimum value of Z occurs at more than two points and is equal to 6.

Q. 31. (a) The probablity that it rains today is 0.4. If it rains today, the probability that it will rain tomorrow is 0.8. If it does not rain today, the probability that it will rain tomorrow is 0.7. If

P_1 : denotes the probability that it does not rain today

P_2 : denotes the probability the it will not rain tomorrow, if it rains today.

P_3 : denotes the probability that it will rain tomorrow, if it does not rain today

P_4 : denotes the probability that it will not rain tomorrow, if it does not rain today.

(i) Find the value of $P_1 \times P_4 - P_2 \times P_3$ [2 Marks]

(ii) Calculate the probability of raining tomorrow. [1 Marks]

Solution. Since the event of raining today and not raining tomorrow are complementary,

∴ If the probability that it rains today = 0.4,

then the probability that it does not rain today

$= 1 - 0.4 = 0.6.$

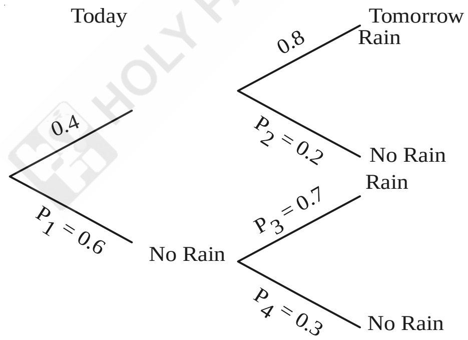
Thus $P_1 = 0.6.$

If it rains today, the probability that it will rain tomorrow is 0.8, then the probability that it will not rain tomorrow

$= 1 - 0.8 = 0.2.$

If it does rain today, the probability that it will rain tomorrow is 0.7, then the probability that it will not rain tomorrow $= 1 - 0.7 = 0.3.$

Thus $P_3 = 0.3.$



1

(*i*) $P_1 \times P_4 - P_2 \times P_3 = 0.6 \times 0.3 - 0.2 \times 0.7$ **1**

$= 0.04.$

(*ii*) Let the events be :

E_1 : It will rain today

E_2 : It will not rain today

A : It will rain-tomorrow

∴ $P(E_1) = 0.4$ and $P(E_2) = 0.6.$

∴ $P(A/E_1) = 0.8$ and $P(A/E_2) = 0.7.$ **½**

Thus, $P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$

$= 0.4 \times 0.8 + 0.6 \times 0.7$

$= 0.32 + 0.42 = 0.74.$

Hence, the probability of rain tomorrow is 0.74. ½

(b) A random variable X can take all non-negative integral values and the probability that X takes the value *r* propartional to 5^{-r}. Find P (X < 3).

We have : $P(X = r) \propto \frac{1}{5^r}$.

$\therefore \quad P(X = r) = k \cdot \frac{1}{5^r}$, when k is non-zero constant ½

$\therefore \quad P(r = 0), \quad k.\frac{1}{5^0}$

$P(r = 1), \quad k.\frac{1}{5^1}$

$P(r = 2), \quad k.\frac{1}{5^2}$

$P(r = 3), \quad k.\frac{1}{5^3}$

....

....

Thus, P (X = 0) + P (X = 1) + P (X = 2) + ... = 1 ½

$$\Rightarrow \quad k\left(1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+...........\right) = 1$$

$$\Rightarrow \quad k\left(\frac{1}{1-\frac{1}{5}}\right) = 1$$

$$\Rightarrow \quad k\left(\frac{5}{4}\right) = 1.$$

Hence, $k = \frac{4}{5}$. ½

$\therefore \quad P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \frac{4}{5}\left(1+\frac{1}{5}+\frac{1}{5^2}\right)$$

$$= \frac{4}{5}\left(\frac{25+5+1}{25}\right)$$

$$= \frac{4}{5}\left(\frac{31}{25}\right) = \frac{124}{125}.$$ **1**

SECTION—D

[5 × 4 = 20]

The section comprises of 4 long answer (LA) type questions of 5 marks each)

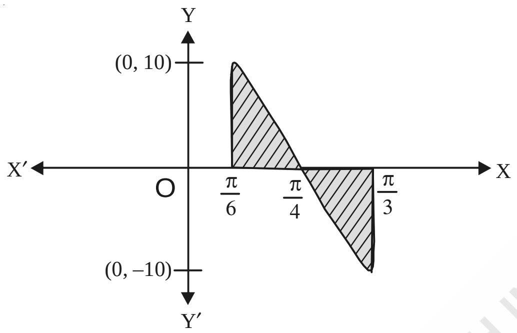
Q. 32. Draw the rough sketch of the curve $y = 20 \cos 2x$; $\left(\text{where } \frac{\pi}{6} \le x \le \frac{\pi}{3}\right)$. Using integration, find the area of the region bounded by the curve $y = 20 \cos 2x$ from the ordinates $x = \frac{\pi}{6}$ to $x = \frac{\pi}{3}$ and the x-axis.

Solution. The given curve is $y = 2x \cos 2x$.

Table :

x :	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y :	10	0	– 10

Graph is as below :



1

$$\text{Reqd. area} = \int_{\pi/6}^{\pi/4} 20\cos 2x \, dx + \left| \int_{\pi/4}^{\pi/3} 20\cos 2x \, dx \right|$$ 1 + 1

$$= 20\left[\frac{\sin 2x}{2}\right]_{\pi/6}^{\pi/4} + \left| 20\left\{\frac{\sin 2x}{2}\right\}_{\pi/4}^{\pi/3} \right| = 10\left(1 - \frac{\sqrt{3}}{2}\right) + 10\left(1 - \frac{\sqrt{3}}{2}\right)$$ 1

$$= 20\left(1 - \frac{\sqrt{3}}{2}\right) \text{ sq. units.}$$ 1

Q. 33. The equation of the path traversed by the ball headed by the footballer is $y = ax^2 + bx + c$; (where $0 \le x \le 14$ and a, b, c $\in$ R and a $\ne$ 0) with respect to a XY-coordinate system in the vertical plane. The ball passes through the points (2, 15), (4, 25) and (14, 15). Determine the values of a, b and c by solving the system of linear equations in a, b and c using matrix method. Also, find the equation of the path traversed by the ball.

Solution. Equation of the path is $y = ax^2 + bx + c$ ...(1)

Since it passes thro' (2, 15), (4, 25) and (14, 15),

$\therefore$ $15 = 4x + 2b + c$ *i.e.* $4a + 2b + c = 15$ ...(2)

$25 = 16a + 4b + c$ *i.e.* $16a + 4b + c = 25$ ...(3)

and $15 = 196a + 14b + c$ *i.e.* $196a + 14b + c = 15$ ...(4) 1

The system of equations can be written as AX = B ...(5),

where $A = \begin{bmatrix} 4 & 2 & 1 \\ 16 & 4 & 1 \\ 196 & 14 & 1 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $B = \begin{bmatrix} 15 \\ 25 \\ 15 \end{bmatrix}$. ½

Now $|A| = \begin{vmatrix} 4 & 2 & 1 \\ 16 & 4 & 1 \\ 196 & 14 & 1 \end{vmatrix}$

$= 4(4-14) - 2(16-196) + 1.(224-784) \quad = -40 + 360 - 560 = -240 \neq 0.$ ½

Thus, A^{-1} exists.

Now adj $A = \begin{bmatrix} -10 & 180 & -560 \\ 12 & -192 & 336 \\ -2 & 12 & -16 \end{bmatrix}' = \begin{bmatrix} -10 & 12 & -2 \\ 180 & -192 & 12 \\ -560 & 336 & -16 \end{bmatrix}$. 1

From (5), $X = A^{-1}B$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\frac{1}{240}\begin{bmatrix} -10 & 12 & -2 \\ 180 & -192 & 12 \\ -560 & 336 & -16 \end{bmatrix}\begin{bmatrix} 15 \\ 25 \\ 15 \end{bmatrix}$$

$$= -\frac{5}{240}\begin{bmatrix} -10 & 12 & -2 \\ 180 & -192 & 12 \\ -560 & 336 & -16 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = -\frac{5}{240}\begin{bmatrix} -30+60-6 \\ -240-960+36 \\ -1680+1680-48 \end{bmatrix}$$

$$= -\frac{5}{240}\begin{bmatrix} 24 \\ -384 \\ -48 \end{bmatrix} = -\frac{1}{48}\begin{bmatrix} 24 \\ -384 \\ -48 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 8 \\ 1 \end{bmatrix}.$$ 1

Thus, $a = -\frac{1}{2}$, $b = 8$ and $c = 1$. ½

Hence, the equation of the path is $y = -\frac{1}{2}x^2 + 8x + 1$. ½

Q. 34 (a) If f : R → R is defined by f (x) = | x |^3, show that f″ (x) exists for all real x and find it.

Solution. We have : $f(x) = |x|^3 = \begin{cases} x^3, \text{if } x \geq 0 \\ (-x)^3 = -x^3 \text{ if } x < 0. \end{cases}$ ½

At x = 0

LHD $= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x^3 - 0}{x} \lim_{x \to 0^-} (-x^2) = 0$ ½

RHD $= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{x^3 - 0}{x} = \lim_{x \to 0^+} (x^2) = 0.$ ½

Thus, **LHD = RHD.** 1

$\therefore$ $f(x)$ is differentiable at $x = 0$ and derivative of $f(x)$ is given by :

$$f'(x) = \begin{cases} 3x^2, \text{ if } x \geq 0 \\ -3x^2, \text{ if } x < 0 \end{cases}.$$ ½

LHD of $f'(x)$ at $x = 0 = \lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0}$ ½

$$= \lim_{x \to 0^-} \frac{-3x^2 - 0}{x} = \lim_{x \to 0^-} (-3x) = 0$$ ½

RHD of $f'(x)$ at $x = 0$

$$= \lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^+} \left(\frac{3x^2 - 0}{x - 2}\right) = \lim_{x \to 0^+} (3x) = 0.$$

Thus **LHD** of $f'(x)$ = **RHD** of $f'(x)$ at $x = 0$. ½

Hence, $f'(x)$ is differentiable at $x = 0$.

Hence, $f''(x) = \begin{cases} 6x, \text{ if } x > 0 \\ -6x, \text{ if } x < 0 \end{cases}.$ ½

Or

(b) If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that : $\dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$ is a constant independent of a and b.

Solution. We have : $(x - a)^2 + (y - b)^2 = c^2$.

Let $x - a = c \cos \theta$ and $y - b = c \sin \theta$. ½

$\therefore$ $\frac{dx}{d\theta} = -c \sin \theta$ and $\frac{dy}{d\theta} = c \cos \theta$. ½

$\therefore$ $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{c\cos\theta}{-c\sin\theta} = -\cot \theta$. 1

Diff. w.r.t. θ. $\frac{d}{d\theta}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}(-\cot\theta)$ ½

$\Rightarrow$ $\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{d\theta} = \operatorname{cosec}^2 \theta$

$\Rightarrow$ $\frac{d^2y}{dx^2}(-c \sin \theta) = \operatorname{cosec}^2 \theta$ ½

$\Rightarrow$ $\frac{d^2y}{dx^2} = -\frac{\operatorname{cosec}^3\theta}{c}$. 1

$$\text{Hence, } \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+\cot^2\theta)^{3/2}}{-\frac{\text{cosec}^3\theta}{c}} = -\frac{c\,\text{cosec}^3\theta}{\text{cosec}^3\theta} = -c,$$

which is constant and is independent of a and b.

Q. 35. (a) Find the shortest distance between the lines l_1 and l_2 whose vector equations are :

$$\vec{r} = (-\hat{i} - \hat{j} - \hat{k}) + \lambda(7\hat{i} - 6\hat{j} + \hat{k}) \text{ and } \vec{r} = (3\hat{i} + 5\hat{j} + 7\hat{k}) + \mu(\hat{i} - 2\hat{j} + \hat{k}),$$

where λ and μ are parameters.

Solution. Comparing given equations with : $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, we have :

$\vec{b_1} = 7\hat{i} - 6\hat{j} + \hat{k}$, $\vec{b_2} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{a_1} = -\hat{i} - \hat{j} - \hat{k}$, $\vec{a_2} = 3\hat{i} + 5\hat{j} + 7\hat{k}$. 1

Now $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \hat{i}(-6+2) - \hat{j}(7-1) + \hat{k}(-14+6) = -4\hat{i} - 6\hat{j} - 8\hat{k}$. 1

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-4)^2 + (-6)^2 + (-8)^2} = \sqrt{16+36+64} = \sqrt{116}$. ½

Also, $\quad \vec{a_2} - \vec{a_1} = (3+1)\hat{i} + (5+1)\hat{j} + (7+1)\hat{k} = 4\hat{i} + 6\hat{j} + 8\hat{k}$.

$\therefore$ d, the shortest distance between the given lines is given by :

$$d = \left|\frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right| = \left|\frac{(-4\hat{i} - 6\hat{j} - 8\hat{k}).(4\hat{i} + 6\hat{j} + 8\hat{k})}{\sqrt{116}}\right|$$ ½

$$= \left|\frac{(-4)(4) + (-6)(6) + (-8)(8)}{\sqrt{116}}\right|$$

$$= \left|\frac{-16-36-64}{\sqrt{116}}\right| = \left|\frac{-116}{\sqrt{116}}\right| = \sqrt{116} = 2\sqrt{29} \text{ units.}$$ ½

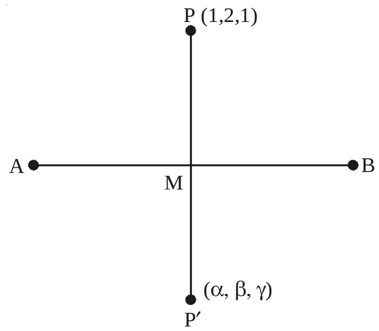
Or

(b) Find the image of the point (1, 2, 1) with respect to the line :

$$\frac{\mathbf{x-3}}{\mathbf{1}} = \frac{\mathbf{y+1}}{\mathbf{2}} = \frac{\mathbf{z-1}}{\mathbf{3}}$$

Also, find the equation of the line joining the given part and its image.

Solution. Let P (1, 2, 1) be the given point.



Let M be the foot of perpendicular from P to the given line AB.

The given line is $\frac{x-3}{1} = \frac{y+1}{2} = \frac{z-1}{3} = \lambda$ (say)

$\therefore$ $x = \lambda + 3, y = 2\lambda - 1, z = 3\lambda + 1.$ ½

Let the co-ordinates of M be (λ + 3), 2λ – 1, 3λ + 1).

$\therefore$ Direction-ratios of PM are

$< \lambda + 3 - 1, 2\lambda - 1 - 2, 3\lambda + 1 - 1 >$

i.e. $< \lambda + 2, 2\lambda - 3, 3\lambda >$

Direction-ratios of AB are $< 1, 2, 3 >$ ½

Since PM ⊥ AB,

$\therefore$ $(\lambda + 2)(1) + (2\lambda - 3)(2) + (3\lambda)(3) = 0$

$\Rightarrow$ $\lambda + 2 + 4\lambda - 6 + 9\lambda = 0$

$\Rightarrow$ $14\lambda = 4 \quad \Rightarrow \lambda = \frac{2}{7}.$ ½

$\therefore$ Co-ordinates of M are $\left(\frac{2}{7}+3, \frac{4}{7}-1, \frac{6}{7}+1\right)$

i.e. $\left(\frac{23}{7}, -\frac{3}{7}, \frac{13}{7}\right).$ ½

Let P′ (α, β, γ) be the image of P (1, 2, 1).

Then M is the mid-point of [PP′] 1

$\therefore$ $\left(\frac{1+\alpha}{2}, \frac{2+\beta}{2}, \frac{1+\gamma}{2}\right) = \left(\frac{23}{7}, -\frac{3}{7}, \frac{13}{7}\right)$

$\therefore$ $\frac{1+\alpha}{2} = \frac{23}{7}, \quad \frac{2+\beta}{2} = -\frac{3}{7}, \quad \frac{1+\gamma}{2} = \frac{13}{7}$

$\Rightarrow$ $1 + \alpha = \frac{46}{7}, \; 2 + \beta = -\frac{6}{7}, \; 1 + \gamma = \frac{26}{7}$

$\Rightarrow$ $\alpha = \frac{39}{7}, \; \beta = -\frac{20}{7}, \; \gamma = \frac{19}{7}.$

Hence, the image of P is P′ $\left(\frac{39}{7}, \frac{-20}{7}, \frac{19}{7}\right).$ 1

Equation of the line PP′ is

$$\frac{x-1}{\frac{39}{7}-1} = \frac{y-2}{-\frac{20}{7}-2} = \frac{z-1}{\left(\frac{19}{7}-1\right)}$$

i.e. $$\frac{x-1}{32/7} = \frac{y-2}{-34/7} = \frac{z-1}{12/7}$$

i.e. $$\frac{x-1}{16} = \frac{y-2}{-17} = \frac{z-1}{6}.$$ 1

SECTION—E

[4 × 3 = 12]

(This section comprises of 3 Case-study/passage-based questions of 4 marks each with subparts. The first two case study questions have three subparts (i), (ii), (iii) of marks 1, 1, 2 respectively. The third case study question has two subparts of 2 marks each)

Case Study–1

Q. 36. Ramesh, the owner of a sweet selling shop, purchased some rectangular card board sheets of dimension 25 cm by 40 cm to make container packets without top. Let *x* cm be the length of the side of the square to be cut out from each corner to give that sheet the shape of the container by folding up the flaps.

Based on the above information answer the following questions :

(i) Express the volume (V) of each container as function of x only. **[1 mark]**

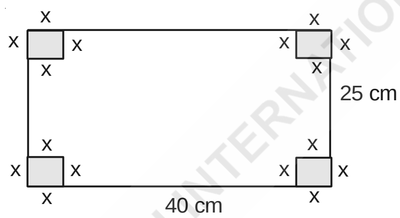
(ii) Find $\frac{dV}{dx}$. **[1 Mark]**

(iii) (a) For what value of x, the volume of each container is maximum ? **[2 Marks]**

OR

(iii) (b) Check whether V has a point of inflection at x = $\frac{65}{6}$ or not ? **[2 Marks]**

Solution. (*i*) V = (40 – 2*x*) (25 – 2*x*) *x*



(*i*) $\frac{dV}{dx} = (40-2x)(25-2x)1$

$(40-2x)(-2)x + (-2)(25-x)x = 4(3x-50)(x-5)$.

(*ii*) For extreme values, $\frac{dV}{dx} = 0$

$\Rightarrow$ 4 (3*x* – 50) (*x* – 5) = 0 $\Rightarrow x = \frac{50}{3}$ or *x* = 5.

$\frac{d^2V}{dx^2} = 24x - 260$

$\therefore \left.\frac{d^2V}{dx^2}\right]_{x=5} = 120 - 260 = -140 < 0$

$\therefore$ V is maximum at *x* = 5.

Or

(*iii*) $\frac{dV}{dx} = 4(3x^2 - 65x + 250)$

$$\frac{d^2V}{dx^2} = 4\,(6x - 65).$$

$$\left.\frac{dV}{dx}\right]_{x=\frac{65}{6}} \text{ exists and } \left.\frac{d^2V}{dx^2}\right]_{x=\frac{65}{2}} = 0$$

$$\left.\frac{d^2V}{dx^2}\right]_{x=\left(\frac{65}{6}\right)^{-1}} \text{ is negative and } \left.\frac{d^2V}{dx^2}\right]_{x=\left(\frac{65}{6}\right)^{+}} \text{ is positive}$$

Hence, $x = \dfrac{65}{6}$ is a point of inflection.

Case Study–2

Q. 37. An organization conducted bike race under 2 different categories-boys and girls. In all, there were 250 participants. Among all of them finally three from Category 1 and two from Category 2 were selected for the final race, Ravi forms two sets B and G with these participants for his college project.

Let B = {(b_1, b_2, b_3}, G = {g_1, g_2}, where B represents the set of boys selected and G the set of girls who were selected for the final race.

Ravi decides to explore these sets for various types of relations and functions :

On the basis of the above information, answer the following questions :

(i) Ravi wishes to form all the relations possible from B to G. How many such relations are possible ? [1 Mark]

(ii) Among these relations, how many are functions from B to G ? [1 Mark]

(iii) (a) Ravi defines a relation from B to B as R_1 = {(b_1, b_2), (b_2, b_1)}. Write the minimum ordered pairs to be added in R_1 so that it becomes (A) reflexive but not symmetric, (B) reflexive and symmetric but not transitive. [2 Marks]

OR

(iii) (b) If the track of the final race (for the biker b_1) follows the curve :

$x^2 = 4y$; (where $0 \le x \le 20\sqrt{2}$ & $0 \le y \le 200$), then state whether the track represents a one-one and onto function or not. (Justify). [2 Marks]

Solution. (*i*) Number of relations

$= $ Number of subsets of the set B × G

$= 2^{n\,(B \times G)} = 2^{n\,(B) \times n\,(G)} = 2^{3 \times 2} = 2^6.$

(*ii*) Number of functions

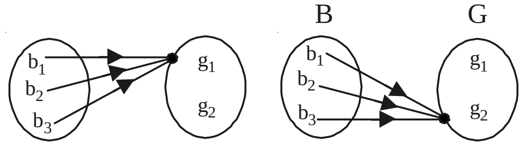
$= $ (Number of elements is co-domain) $= [n\,(G)^{n(B)}] = 2^3.$

(*iii*) (*a*) Number of one-one functions = 0

[$\because$ number of elements in co-domain (G)] < number elements in domain (B)

Number of onto function = $2^3 - 2 = 6$

Total number of functions = $[n\,(G)]^{n(B)} = 2^3.$



Hence, the number of onto functions

Or

(*b*) One-one and onto function $x^2 = 4y$.

Let $y = f(x) = \dfrac{x^2}{4}$.

Let $x_1, x_2 \in [0, 20\sqrt{2}]$

$$f(x_1) = f(x_2) = \frac{x_1^2}{4} = \frac{x_2^2}{4}$$

$\Rightarrow \quad x_1^2 = x_2^2 \Rightarrow (x_1^2 - x_2^2) = 0$

$\Rightarrow \quad (x_1 - x_2)(x_1 + x_2) = 0 \Rightarrow x_1 = x_2$ $\qquad [\because x_1, x_2 \in [0, 20\sqrt{2}]]$

Hence, f is one-one

Now $0 \le 4 \le 200 \Rightarrow y$ is non-negative and $f(2\sqrt{y}) = y$.

For any $y \in [0, 2]$, the pre image of y is $[0, 20\sqrt{2}]$. Hence, f is onto

Case-Study-3

Q. 38. Arka bought two cages of birds : Cage-I contains 5 parrots and 1 owl and Cage-II contains 6 parrots. One day Arka forgot to lock both cages and two birds flew from Cage-I to Cage-II (simultaneously). Then two birds flew back from cage-II to cage-I (simultaneously).

Assume that all the birds have equal chances of flying.

On the basis of the above information, answer the following questions :

(i) When two birds flew from Cage-I to Cage-II and two birds flew back from Cage-II to Cage-I, then find the probability that the owl is still in Cage-I. [2 Marks]

(ii) When two birds flew frm Cage-I to Cage-II and two birds flew back from Cage-II to Cage-I, the owl is still seen in Cage-I, what is the probabilitythat one parrot and the owl flew from Cage-I to Cage-II ? [2 Marks]

Solution. Let the events be :

E_1 : Parrot and owl fly from cage I

E_2 : Owl still in cage-I

E_2^c : Owl is not in cage-I

$$n(E_1 \cap E_3) = (^5C_1 \times {}^1C_1)(^7C_1 \times {}^1C_1) = (5 \times 1)(7 \times 1) = 35$$

$$n(E_1 \cap E_3^c) = (^5C_1 \times {}^1C_1)(^7C_2) = (5 \times 1)(21) = 105$$

$$n(E_2 \cap E_3) = (^5C_2)(^8C_2) = (10)(28) = 280$$

$$n(E_2 \cap E_3^c) = 0$$

$\therefore \qquad n(S) = 35 + 105 + 280 = 420.$

(*i*) Probability that Owl is still in cage-I

$$= P(E_3) = P(E_1 \cap E_3) + P(E_2 \cap E_3)$$

$$= \frac{35 + 280}{420} = \frac{315}{420} = \frac{3}{4}.$$

(*ii*) Probability that one parrot and owl flew from cage I to cage II, given then the Owl is still in cage I = $P(E_1/E_3)$

$$P(E_1/E_3) = \frac{P(E_1 \cap E_3)}{P(E_3)} = \frac{P(E_1 \cap E_3)}{P(E_1 \cap E_3) + P(E_2 \cap E_3)} \qquad [\text{Baye's Theorem}]$$

$$= \frac{\frac{35}{420}}{\frac{315}{420}} = \frac{35}{315} = \frac{1}{9}.$$

MODEL ANSWERS
(Sample Papers 1—11)

Holy Faith New Style Sample Paper–1
(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions :

Read the following instructions very carefully and strictly follow them :

(*i*) *This question paper contains **38** questions. **All** questions are **compulsory.***
(*ii*) *This question paper is divided into **five** Sections–**A, B, C, D** and **E.***
(*iii*) *In **Section-A,** Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.*
(*iv*) *In **Section-B,** Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.*
(*v*) *In **Section-C,** Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.*
(*vi*) *In **Section–D,** Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.*
(*vii*) *In **Section-E,** Questions no. **36** to **38** are **Case Study-based** questions carrying **4** marks each.*
(*viii*) *There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.*
(*ix*) *Use of calculators is **not** allowed.*

SECTION—A

(Multiple choice questions, each question carries 1 mark)

Q. 1. If the set A contains 5 elements and the set B contains 6 elements, then the number of both one-one and onto mappings from A to B is :

(*a*) 720 (*b*) 120
(*c*) 30 (*d*) 0.

Ans. (*d*) 0

Q. 2. $\sin\left[\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right]$ **is equal to :**

(*a*) 1 (*b*) $\frac{1}{2}$
(*c*) $\frac{1}{3}$ (*d*) $\frac{1}{4}$.

Ans. (*a*) 1

Q. 3. The value of $x - y + z$ from the following equation :

$$\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix} \text{ is :}$$

(*a*) – 3 (*b*) – 1
(*c*) 1 (*d*) 3.

Ans. (*c*) 1

Q. 4. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ **and A. adj A = A A^T, then $5a + b$ is equal to :**

(*a*) 5 (*b*) 4
(*c*) 13 (*d*) – 1.

Ans. (*a*) 5

Q. 5. If A be a 3 × 3 square matrix such that :

$$A\,(\text{adj. } A) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

then the value of | adj. A | is :

(*a*) 5 (*b*) 25
(*c*) 125 (*d*) 625.

Ans. (*b*) 25

Q. 6. If $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ **is continuous at $x = 0$, then the value of 'k' is :**

(*a*) –3 (*b*) 0
(*c*) 3 (*d*) any real number.

Ans. (*a*) –3

Q. 7. If $y = \sin(m \sin^{-1} x)$, then which one of the following equations is true ?

(*a*) $(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + m^2y = 0$

(*b*) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

(*c*) $(1+x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

(*d*) $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2x = 0$.

Ans. (*b*) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

Q. 8. In the interval (1, 2), the function :

$f(x) = 2\,|x-1| + 3\,|x-2|$ is :

(*a*) Strictly increasing
(*b*) Strictly decreasing
(*c*) Neither increasing nor decreasing
(*d*) Remains constant.

Ans. (*b*) Strictly decreasing

Q. 9. $\int \frac{dx}{\sin^2 x \cos^2 x}$ **equals :**

(*a*) $\tan x + \cot x + c$ (*b*) $\tan x - \cot x + c$
(*c*) $\tan x \cot x + c$ (*d*) $\tan x - \cot 2x + c$.

Ans. (*b*) $\tan x - \cot x + c$

Q. 10. The value of $\int_0^{\pi/4} \sec 2x\, dx$ **is:**

(*a*) 0 (*b*) 1
(*c*) $\frac{1}{2}$ (*d*) $-\frac{1}{2}$.

Ans. (*b*) 1

Q. 11. Area of the region bounded by the curve $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant is :

(*a*) $4(4-\sqrt{2})$ sq. units
(*b*) $(4-\sqrt{2})$ sq. units
(*c*) $(16-\sqrt{2})$ sq. units
(*d*) $4\sqrt{2}$ sq. units.

Ans. (*a*) $4(4-\sqrt{2})$ sq. units

Q. 12. The degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2}$ **is :**

(*a*) 4 (*b*) $\frac{3}{2}$
(*c*) 2 (*d*) Not defined.

Ans. (*c*) 2

Q. 13. The general solution of the differential equation : $ydx - xdy = 0$; (Given $x, y > 0$) is of the form :

(*a*) $xy = c$ (*b*) $x = cy^2$
(*c*) $y = cx$ (*d*) $y = cx^2$.

Ans. (*c*) $y = cx$

Q. 14. ABCD is a rhombus whose diagonals intersect at E. Then $\overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED}$ equals to :

(*a*) $\vec{0}$ (*b*) $\overrightarrow{AD}$
(*c*) $2\overrightarrow{BD}$ (*d*) $2\overrightarrow{AD}$.

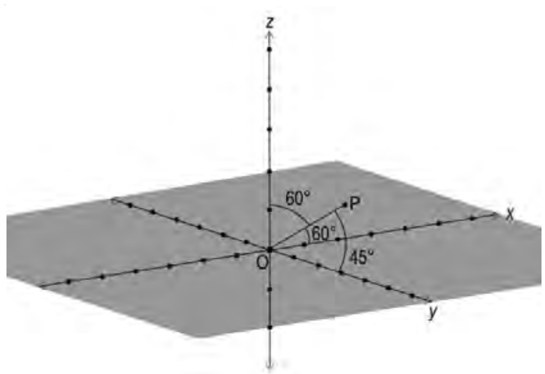
Ans. (*a*) $\vec{0}$

Q. 15. If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ a non-zero scalar, then $\lambda \vec{a}$ is unit vector if :

(*a*) $\lambda = 1$ (*b*) $\lambda = -1$
(*c*) $a = |\lambda|$ (*d*) $a = \frac{1}{|\lambda|}$.

Ans. (*d*) $a = \frac{1}{|\lambda|}$.

Q. 16. A line $\overrightarrow{OP}$ in space, represented by the figure given below, has a magnitude of $2\sqrt{2}$ units.



Which of these are the direction-ratios of $\overrightarrow{OP}$?

(*a*) $(2, \sqrt{2}, 2)$ (*b*) $(\sqrt{2}, 2, \sqrt{2})$
(*c*) $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ (*d*) $(2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2})$.

Ans. (*b*) $(\sqrt{2}, 2, \sqrt{2})$

Q. 17.The solution set of the in equation $3x + 2y > 3$ is :

(*a*) Half plane containing the origin
(*b*) Half plane not containing the origin
(*c*) The point being on the line $3x + 2y = 3$
(*d*) None of these.

Ans. (*b*) Half plane not containing the origin

Q. 18. A problem in Mathematics is given to three students whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. If the events of their solving the problem are independent, then the probability that the problem will be solved is :

(*a*) $\frac{1}{4}$ (*b*) $\frac{1}{3}$

(*c*) $\frac{1}{2}$ (*d*) $\frac{3}{4}$.

Ans. (*d*) $\frac{3}{4}$

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.

(*b*) Both 'A' and 'R' are true but 'R' is not the correct explanation of 'A'.

(*c*) 'A' is true but 'R' is false.

(*d*) 'A' is false but 'R' is true.

Q. 19. X = {0, 2, 4, 6, 8}.

P is a relation on X defined by :

P = {(0, 2), (4, 2), (4, 6), (8, 6), (2, 4), (0, 4)}.

Based on the above information, two statement one given below :

Assertion (A) : The relation P an set X is a transtion relation.

Reason (R) : The relation P has a rubset of the from

{(*a*, *b*), (*b*, *c*), (*a*, *c*)}, what *a*, *b*, *c*, ∈X.`

Ans. (*d*), **Assertion** (A) is false

Reason (R) is true.

Q. 20. Statement–(A) : The differential equation :

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$ cannot be solved by putting $x = vy$.

Statement–(R) : The given equation $\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$ is homogeneous and the substitution for solving is $y = vx$.

Ans. (*d*) The equation $\frac{dy}{dx} = \frac{x^2 y^2}{2xy}$ can be solved by putting $x = vy$.

We can also solve it by putting $y = vx$.

Thus (A) is false and (R) is true.

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Find the domain of the function :
$y = \cos^{-1} (| x - 1 |)$.
Show your steps.

Ans. Since the domain of $\cos^{-1}\theta$ is [–1, 1],

∴ the domain of $y = \cos^{-1}(|x-1|)$ is as below :

$$-1 \le x-1 \le 1 \quad \text{and} \quad -1 \le 1-x \le 1$$
$$\Rightarrow \quad 0 \le x \le 2 \quad \text{and} \quad 1 \ge x-1 \ge -1$$
$$\textit{i.e.} \quad 2 \ge x \ge 0 .$$

Hence, the domain of $\cos^{-1}(|x-1|)$ is [0, 2].

Or

Prove that $3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$; $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Ans. Put $\sin^{-1} x = \theta$ so that $x = \sin\theta$.

$$\textbf{RHS} = \sin^{-1}(3 \sin\theta - 4 \sin^3\theta)$$
$$= \sin^{-1}(\sin 3\theta) = 3\theta$$
$$= 3 \sin^{-1} x = \textbf{LHS}.$$

Q. 22. The population of rabbits in a forest is modelled by the function below :

$P(t) = \frac{2000}{1 + e^{-0.5t}}$, where P represents the population of rabbits in *t* years. Determine whether the rabbit population is increasing or not, and justify your answer.

Ans. We have :

$$P(t) = \frac{2000}{1 + e^{-0.5t}}.$$

$$\therefore \quad P'(t) = 2000 \times \frac{(-1)}{(1+e^{-0.5t})^2} \times e^{-0.5t} \times \left(-\frac{1}{2}\right)$$

$$= \frac{1000\,(e^{-0.5t})}{(1+e^{-0.5t})^2} > 0 \text{ for any value of } t.$$

Hence, the rabbit population is increasing.

Q. 23. Find the absolute maximum and the absolute minimum value of the function given by :

$f(x) = \sin^2 x - \cos x, x \in [0, \pi]$.

Ans. We have :

$$f(x) = \sin^2 x - \cos x.$$
$$\therefore \quad f'(x) = 2 \sin x \cos x + \sin x$$
$$= \sin x\,(2 \cos x + 1).$$
$$\text{Now} \quad f'(x) = 0 \Rightarrow \sin x\,(2 \cos x + 1) = 0$$
$$\Rightarrow \quad \sin x = 0 \quad \text{or} \cos x = -\frac{1}{2}.$$
$$\Rightarrow \quad x = 0, \pi \text{ or } x = \frac{2\pi}{3}.$$
$$\text{Now} \quad f(0) = 0 - 1 = -1$$

$$f\left(\frac{2\pi}{3}\right) = \sin^2\left(\frac{2\pi}{3}\right) - \cos\frac{2\pi}{3}$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

and $f(\pi) = \sin^2 \pi - \cos \pi = (0)^2 - (-1) = 1.$

Hence, absolute maximum value is $\frac{5}{4}$ and absolute minimum value is – 1.

Or

Find two positive numbers whose sum is 24 and their sum of squares is minimum.

Ans. Let 'x' and 'y' be two positive numbers.

By the question, $x + y = 24$ ...(1)

Let $S = x^2 + y^2$

$\Rightarrow$ $S = x^2 + (24 - x)^2$ [*Using (1)*]

$= x^2 + 576 + x^2 - 48x$

$= 2x^2 - 48x + 576.$

$\therefore$ $\frac{dS}{dx} = 4x - 48.$

For S to be minimum, $\frac{dS}{dx} = 0$, which gives :

$4x - 48 = 0 \Rightarrow 4x = 48 \Rightarrow x = 12.$

Now $\frac{d^2S}{dx^2} = 4$, which is +ve for $x = 12$.

Hence, S is minimum when the two positive numbers are 12 and (24 – 12) *i.e.* 12 and 12.

Q. 24. Evaluate : $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$.

Ans. Let $I = \displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ ...(1)

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

[* *Property V*]

$\Rightarrow$ $I = \displaystyle\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$ ...(2)

Adding (1) and (2),

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} 1 \,.\, dx$$

$$= [x]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}.$$

Hence, $I = \frac{\pi}{4}$.

Q. 25. Discuss the continuity of the function :
$f(x) = |x|$ at $x = 0$.

Ans. By *definition*, $f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0. \end{cases}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = \lim_{h \to 0} (-(0 - h)) = \lim_{h \to 0} (h)$$

$= 0.$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = \lim_{h \to 0} (0 + h) = \lim_{h \to 0} (h)$$

$= 0.$

Also, $f(0) = 0.$

Thus $\displaystyle\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$. [$\because$ *Each = 0*]

Hence, 'f' is continuous at $x = 0$.

SECTION—C

(This section comprises of short answer type questions (SA) of 3 marks each)

Q. 26. Integrate : $\displaystyle\int \cos^3 x \sin^4 x \, dx$.

Ans. Since the index of cos x is 3 *i.e.* positive odd integer,

$\therefore$ **Put sin x = t** (*See Above Rule*)

so that $\cos x \, dx = dt$.

$$\therefore \int \cos^3 x \sin^4 x \, dx = \int \cos^2 x \,.\, \cos x \,.\, \sin^4 x \, dx$$

$$= \int \cos^2 x \,.\, \sin^4 x \,.\, \cos x \, dx$$

$$= \int (1 - \sin^2 x) \sin^4 x \,.\, \cos x \, dx$$

$$= \int (1 - t^2)\, t^4 \, dt = \int (t^4 - t^6)\, dt$$

$$= \frac{1}{5}t^5 - \frac{1}{7}t^7 + c = \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c.$$

Q. 27. Consider the experiment of tossing a coin. If the coin shows head, toss it again but if it shows tail, then throw a die. Find the conditional probability of the event that 'the die shows a number greater than 4', given that 'there is atleast one tail'.

Ans. We represent the outcomes of the experiment in the following diagram :

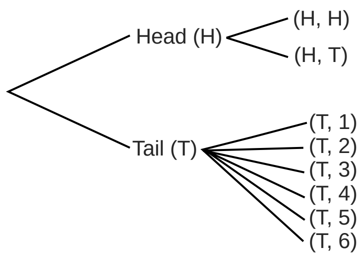


Fig.

Thus the sample space is :

S = {(H, H), (H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)},

where H ≡ Head and Tail ≡ Tail and (H, H) denotes both the tosses result into head and (T, *i*) denotes the first toss results into a tail and the number *i* appears on the die for *i* = 1, 2, 3, 4, 5, 6.

∴The probabilities of these 8 elementary events :

(H, H), (H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)

are $\frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}$ respectively.

These are exhibited in the following diagram :

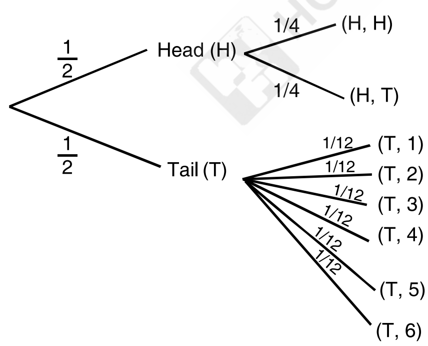


Fig.

Let the events be :

E : 'die shows a number greater than 4'

and F : 'Atleast one tail'

i.e. E = {(T, 5), (T, 6)}

and F = {(H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)}

Also, E ∩ F = {(T, 5), (T, 6)}.

Now P (F) = P ({HT}) + P ({T, 1}) + P ({T, 2}) + P ({T, 3}) + P ({T, 4}) + ({T, 5}) + P ({T, 6})

$$= \frac{1}{4}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}=\frac{3}{4}$$

and P (E ∩ F) = P ({T, 5}) + P ({T, 6})

$$= \frac{1}{12}+\frac{1}{12}$$

$$= \frac{1}{6}.$$

Hence, $P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{1/6}{3/4} = \frac{2}{9}.$

Q. 28. Find : $\int e^x \frac{\sqrt{1+\sin 2x}}{1+\cos 2x} dx.$

Ans. Let $I = \int e^x \frac{\sqrt{1+\sin 2x}}{1+\cos 2x} dx$

$$= \int e^x \frac{\sqrt{\cos^2 x+\sin^2 x+2\cos x\sin x}}{1+\cos 2x} dx$$

$$= \int e^x \frac{\sqrt{(\cos x+\sin x)^2}}{2\cos^2 x} dx$$

$$= \frac{1}{2}\int e^x \left(\frac{\cos x+\sin x}{\cos^2 x}\right) dx$$

$$= \frac{1}{2}\int e^x \left(\frac{1}{\cos x}+\frac{1}{\cos x}.\frac{\sin x}{\cos x}\right) dx$$

$$= \frac{1}{2}\int e^x (\sec x+\sec x \tan x) dx$$

I"**Form :** $\int e^x (f(x)+f'(x))\, dx$"

$$= \frac{1}{2}\left[\int \sec x . e^x \, dx+\int \sec x \tan x . e^x dx\right]$$

$$= \frac{1}{2}\left[\sec x . e^x - \int \sec x \tan x . e^x dx+\int \sec x \tan x . e^x dx\right]$$

[*Integrating first integral by parts*]

$$= \frac{1}{2}[\sec x \,.\, e^x] + c$$

$$= \frac{1}{2}e^x \sec x + c.$$

Or

Evaluate : $\displaystyle\int_{-1}^{1} \frac{x + |x| + 1}{x^2 + 2|x| + 1}\,dx.$

Ans. $\displaystyle I = \int_{-1}^{1} \frac{x + |x| + 1}{x^2 + 2|x| + 1}\,dx$

$$= \int_{-1}^{1} \frac{x}{x^2 + 2|x| + 1}\,dx + \int_{-1}^{1} \frac{|x| + 1}{x^2 + 2|x| + 1}\,dx$$

$$= I_1 + I_2 \text{ (say)} \qquad \text{...(1)}$$

Now, $\displaystyle I_1 = \int_{-1}^{1} \frac{x}{x^2 + 2|x| + 1}\,dx.$

Let $f(x) = \dfrac{x}{x^2 + 2|x| + 1}.$

$$\therefore \quad f(-x) = \frac{-x}{(-x)^2 + 2|-x| + 1} = \frac{-x}{x^2 + 2|x| + 1} = -f(x)$$

$\Rightarrow f(x)$ is an odd function.

Thus, $I_1 = 0$ ...(2)

And, $\displaystyle I_2 = \int_{-1}^{1} \frac{|x| + 1}{x^2 + 2|x| + 1}\,dx.$

Let $g(x) = \dfrac{|x| + 1}{x^2 + 2|x| + 1}.$

$$\therefore \quad g(-x) = \frac{|-x| + 1}{(-x)^2 + 2|-x| + 1} = \frac{|x| + 1}{x^2 + 2|x| + 1} = g(x)$$

$\Rightarrow g(x)$ is an even function.

$$\therefore \quad I_2 = 2\int_0^1 \frac{x + 1}{x^2 + 2x + 1}\,dx \qquad [\because x > 0]$$

$$= 2\int_0^1 \frac{x + 1}{(x + 1)^2}\,dx = 2\int_0^1 \frac{1}{x + 1}\,dx$$

$$= 2\,[\log(x + 1)]_0^1 = 2\,[\log 2 - \log 1]$$

$$= 2\,[\log 2 - 0] = 2\log 2 \qquad \text{...(3)}$$

From (1), (2) and (3), I = 0 + 2 log 2 = 2 log 2.

Q. 29. Show that $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$ *or*

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy}$$

is homogeneous and solve it.

Solution. The given equation is :

$$(x^2 + xy)\,dy = (x^2 + y^2)\,dx$$

i.e. $\displaystyle \frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy}$ ...(1)

Here $f(x, y) = \dfrac{x^2 + y^2}{x^2 + xy}.$

$$\therefore \quad f(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + \lambda^2 y^2}{\lambda^2 x^2 + \lambda x \,.\, \lambda y} = \frac{\lambda^2 (x^2 + y^2)}{\lambda^2 (x^2 + xy)}$$

$$= \lambda^0 \frac{x^2 + y^2}{x^2 + xy} = \lambda^0 f(x, y).$$

Thus $f(x, y)$ is homogeneous function of degree zero.

Put y = vx ...(2)

so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ ...(3)

From (1), using (2) and (3), we get :

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{x^2 + vx^2} = \frac{1 + v^2}{1 + v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 + v^2}{1 + v} - v$$

$$= \frac{1 + v^2 - v - v^2}{1 + v} = \frac{1 - v}{1 + v}$$

$$\Rightarrow \quad \frac{1 + v}{1 - v}\,dv = \frac{dx}{x}. \qquad |\textit{Variables Separable}$$

Integrating, $\displaystyle \int \frac{dx}{x} = \int \frac{1 + v}{1 - v}\,dv = \int \frac{2 - (1 - v)}{1 - v}\,dv$

$$= 2\int \frac{1}{1 - v}\,dv - \int 1 \,.\, dv$$

$$= -2\int \frac{-1}{1 - v}\,dv - \int 1 . dv$$

$$\Rightarrow \quad \log|x| = -2\log|1 - v| - v + c$$

$$\Rightarrow \quad \log|x| = -2\log\left|1 - \frac{y}{x}\right| - \frac{y}{x} + c$$

$$\Rightarrow \quad \log|x| = -2\log\left|\frac{x - y}{x}\right| - \frac{y}{x} + c$$

$$\Rightarrow \quad \log|x| = -2\log|x - y| + 2\log|x| - \frac{y}{x} + c$$

$\Rightarrow \log|x| - 2\log|x-y| - \frac{y}{x} + c = 0,$

which is the reqd. solution.

Or

Find the general solution of the following differential equation :

$$\mathbf{(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx} = 0.}$$

Ans. The given equation is :

$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x-e^{\tan^{-1}y})\frac{dy}{dx} = -(1+y^2)$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x-e^{\tan^{-1}y}}{-(1+y^2)}$$

$$\Rightarrow \quad \frac{dx}{dy}+\frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2} \qquad ...(1)$$

Comparing with $\frac{dx}{dy} + \text{P}x = \text{Q}$, we have :

$$\text{'P'} = \frac{1}{1+y^2} \text{ and 'Q'} = \frac{e^{\tan^{-1}y}}{1+y^2}.$$

$$\therefore \text{ I.F.} = e^{\int \text{P}\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}.$$

Multiplying (1) with $e^{\tan^{-1}y}$, we get :

$$e^{\tan^{-1}y}.\frac{dx}{dy}+\frac{x}{1+y^2}e^{\tan^{-1}y} = \frac{(e^{\tan^{-1}y})^2}{1+y^2}$$

$$\Rightarrow \quad \frac{d}{dy}\left(xe^{\tan^{-1}y}\right) = \frac{(e^{\tan^{-1}y})^2}{1+y^2}.$$

Integrating, $xe^{\tan^{-1}y} = \int \frac{(e^{\tan^{-1}y})^2}{1+y^2}dy + c \qquad ...(2)$

Now $\text{I} = \int \frac{(e^{\tan^{-1}y})^2}{1+y^2}dy.$

Put $\tan^{-1} y = t$ so that $\frac{1}{1+y^2}dy = dt.$

$$\therefore \text{ I} = \int (e^t)^2\,dt = \int e^{2t}\,dt = \frac{e^{2t}}{2} = \frac{e^{2\tan^{-1}y}}{2}.$$

Putting in (2), $xe^{\tan^{-1}y} = \frac{1}{2}e^{2\tan^{-1}y} + c$

$$\Rightarrow x = \frac{1}{2}e^{\tan^{-1}y} + c\,e^{-\tan^{-1}y} + c,$$

which is the reqd. solution.

30. Maximize Z = 5x + 3y

subject to the constraints :

3x + 5y ≤ 15, 5x + 2y ≤ 10, x ≥ 0, y ≥ 0.

Ans.

Solution. The system of constraints is :

$3x + 5y \leq 15$ ...(1)

$5x + 2y \leq 10$ ...(2)

and $x \geq 0,\ y \geq 0$ ...(3)

The shaded region in the following figure is the feasible region determined by the system of constraints (1) – (3) :

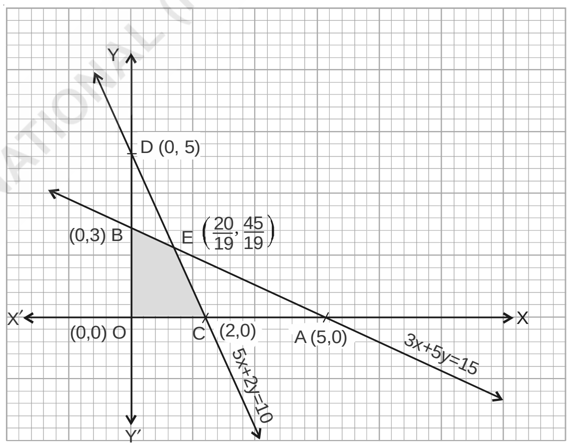


Fig.

It is observed that the feasible region OCEB is bounded. Thus we use ***Corner Point Method*** to determine the maximum value of Z, where :

$$Z = 5x + 3y \qquad ...(4)$$

The co-ordinates of O, C, E and B are (0, 0), (2, 0), $\left(\frac{20}{19}, \frac{45}{19}\right)$ (*Solving 3x + 5y = 15 and 5x + 2y = 10*) and (0, 3) respectively.

We evaluate Z at each corner point :

Corner Point	Corresponding Value of Z
O : (0, 0)	0
C : (2, 0)	10
E : $\left(\frac{20}{19}, \frac{45}{19}\right)$	$\mathbf{\frac{235}{19}}$ **(Maximum)**
B : (0, 3)	9

Hence, $Z_{max} = \frac{235}{19}$ at the point $\left(\frac{20}{19}, \frac{45}{19}\right)$.

Or

Minimize Z = 3*x* + 2*y*

subject to the constraints :

***x* + *y* ≥ 8, 3*x* + 5*y* ≤ 15, *x* ≥ 0, *y* ≥ 0.**

Solution. The system of constraints is :

$x + y \geq 8$ ...(1)

$3x + 5y \leq 15$ ...(2)

and $x, y \geq 0$ ...(3)

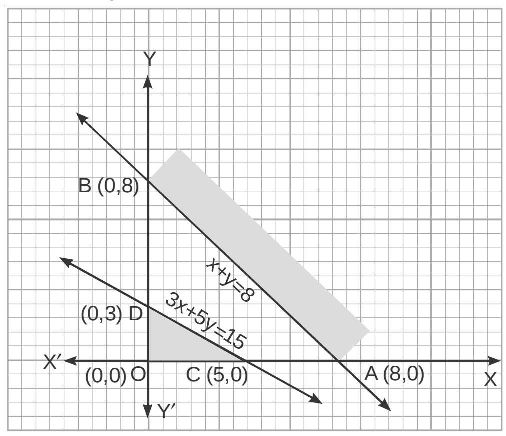


Fig.

It is observed that there is no point, which satisfies all (1) – (3) simultaneously.

Thus there is no feasible region.

Hence, there is no feasible solution.

Q. 31. Maximize Z = 5*x* + 3*y*

subject to the constraints :

3*x* + 5*y* ≤ 15, 5*x* + 2*y* ≤ 10, *x* ≥ 0, *y* ≥ 0.

Ans. The system of constraints is :

$3x + 5y \leq 15$ ...(1)

$5x + 2y \leq 10$ ...(2)

and $x \geq 0, y \geq 0$ ...(3)

The shaded region in the following figure is the feasible region determined by the system of constraints (1) – (3) :

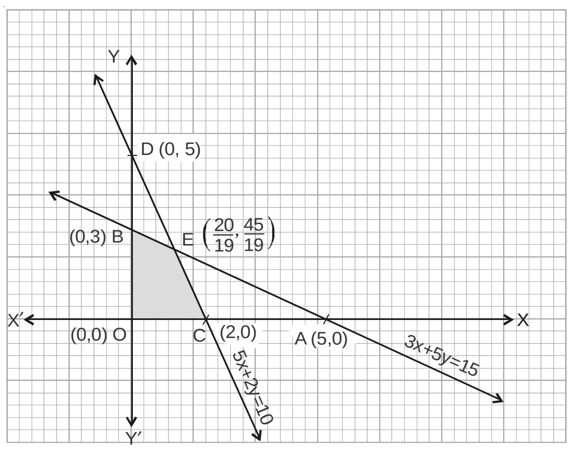


Fig.

It is observed that the feasible region OCEB is bounded. Thus we use ***Corner Point Method*** to determine the maximum value of Z, where :

$Z = 5x + 3y$ ...(4)

The co-ordinates of O, C, E and B are (0, 0), (2, 0), $\left(\frac{20}{19}, \frac{45}{19}\right)$ (*Solving 3x + 5y = 15 and 5x + 2y = 10*) and (0, 3) respectively.

We evaluate Z at each corner point :

Corner Point	Corresponding Value of Z
O : (0, 0)	0
C : (2, 0)	10
E : $\left(\frac{20}{19}, \frac{45}{19}\right)$	$\mathbf{\frac{235}{19}}$ **(Maximum)**
B : (0, 3)	9

Hence, $Z_{max} = \frac{235}{19}$ at the point $\left(\frac{20}{19}, \frac{45}{19}\right)$.

SECTION—D

(This section comprises of Long Answer Type questions (LAQ) of 5 marks each)

Example 32. Find the area enclosed by the circle : $\mathbf{x^2 + y^2 = a^2}$.

Solution. The given circle is $x^2 + y^2 = a^2$ ....(1)

This is a circle whose centre is (0, 0) and radius '*a*'.

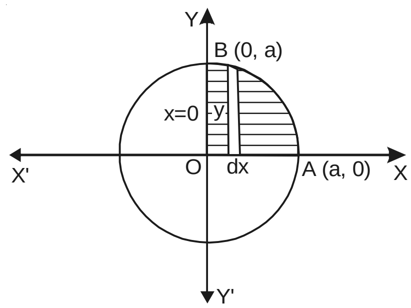


Fig.

Area of the circle = 4 (area of the region OABO, bounded by the curve, *x*-axis and ordinates $x = 0$, $x = a$)

[∵ *Circle is symmetrical about both the axes*]

$$= 4\int_0^a y\,dx \qquad [\textit{Taking vertical strips}]$$

$$= 4\int_0^a \sqrt{a^2 - x^2}\,dx \qquad [\because (1) \Rightarrow y = \pm\sqrt{a^2 - x^2}.$$

But region OABO lies in Ist quadrant, ∴ y is + ve]

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left\{\frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1)\right\} - \{0+0\}\right]$$

$$= 4\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right) = \pi a^2 \text{ sq. units.}$$

Q. 33. Let R be the relation in the set Z of intgers given by $R = \{(a, b) : 2 \text{ divides } a - b\}$. Show that the relation R is transitive. Write the equivalence relation [0].

Ans. (*i*) Let 2 divide $a - b$ and 2 divide $b - c$, where $a, b, c \in \mathbf{Z}$.

$\therefore$ 2 divides $[(a - b) + (b - c)]$

$\Rightarrow$ 2 divides $(a - c)$.

Hence, R is transitive.

(*ii*) $[0] = \{0, \pm 2, \pm 4, \pm 6, ...\}$.

Or

Prove that the function f : R → R given by:

f (x) = 2x is one–one and onto.

Ans.

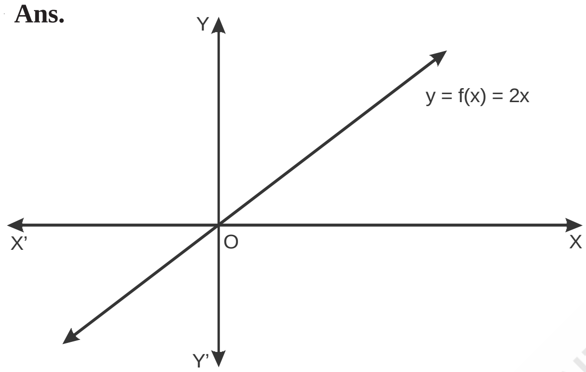


Fig.

Let $x_1, x_2 \in \mathbf{R}$.

Now $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

$\Rightarrow$ 'f' is one–one.

Let $y \in \mathbf{R}$. Let $y = f(x_0)$.

Then $2x_0 = y \Rightarrow x_0 = \frac{y}{2}$.

Now $y \in \mathbf{R} \Rightarrow \frac{y}{2} \in \mathbf{R} \Rightarrow x_0 \in \mathbf{R}$.

$f(x_0) = 2x_0 = 2\left(\frac{y}{2}\right) = y$.

$\therefore$ For each $y \in \mathbf{R}$, there exists $x_0 \in \mathbf{R}$ such that $f(x_0) = y$.

$\therefore$ 'f' is onto.

Hence, 'f' is one–one and onto.

Q. 37. Show that the points :

A $(2\hat{i} - \hat{j} + \hat{k})$, B $(\hat{i} - 3\hat{j} - 5\hat{k})$, C $(3\hat{i} - 4\hat{j} - 4\hat{k})$ are the vertices of a right-angled triangle.

Ans. Let O be the origin of reference.

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$$

$$= -\hat{i} - 2\hat{j} - 6\hat{k},$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})$$

$$= 2\hat{i} - \hat{j} + \hat{k}$$

and
$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$$

$$= (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$$

$$= -\hat{i} + 3\hat{j} + 5\hat{k}.$$

Now
$$|\overrightarrow{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2}$$

$$= \sqrt{1 + 4 + 36} = \sqrt{41},$$

$$|\overrightarrow{BC}| = \sqrt{(2)^2 + (-1)^2 + (1)^2}$$

$$= \sqrt{4 + 1 + 1} = \sqrt{6}$$

and
$$|\overrightarrow{CA}| = \sqrt{(-1)^2 + (3)^2 + (5)^2}$$

$$= \sqrt{1 + 9 + 25} = \sqrt{35}.$$

Thus $AB^2 = BC^2 + CA^2$. [$\because$ *41 = 6 + 35*]

Hence, the triangle is a right-angled triangle.

Or

Find a vector of magnitude 5 units, and parallel to the resultant of the vectors :

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Ans. We have : $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$

and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

$\therefore$ Resultant of vectors $\vec{a}$ and $\vec{b}$ = $\vec{a} + \vec{b}$

$$= (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 2\hat{j} + \hat{k})$$

$$= 3\hat{i} + \hat{j}.$$

$\therefore$ Reqd. vector of magnitude 5 units and parallel to $(\vec{a} + \vec{b})$

$$= 5\frac{(\vec{a} + \vec{b})}{|\vec{a} + \vec{b}|} = 5\frac{3\hat{i} + \hat{j}}{\sqrt{9 + 1 + 0}}$$

$$= \frac{5}{\sqrt{10}}(3\hat{i} + \hat{j}) = \frac{15}{\sqrt{10}}\hat{i} + \frac{5}{\sqrt{10}}\hat{j}.$$

Q. 35. Three persons A, B and C apply for a job of manager in a Private Company. Chances of their selection (A, B and C) are in the ratio 1 : 2 : 4. The probabilities that A, B and C can introduce changes to improve profits of the company are 0·8, 0·5 and 0·3 respectively. If the change does not take place, find the probability that it is due to the appointment of C.

Ans. Let 'E' be the event when change takes place.

Now, we have : $P(A) = \frac{1}{7}$, $P(B) = \frac{2}{7}$ and $P(C) = \frac{4}{7}$ and

$P(E/A) = 1 - 0{\cdot}8 = 0{\cdot}2$, $P(E/B) = 1 - 0{\cdot}5 = 0{\cdot}5$ and $P(E/C) = 1 - 0{\cdot}3 = 0{\cdot}7$.

By *Bayes' Theorem,*

$$P(C/E) = \frac{P(C)\,P(E/C)}{P(A)\,P(E/A) + P(B)\,P(E/B) + P(C)\,P(E/C)}$$

$$= \frac{\left(\frac{4}{7}\right)(0{\cdot}7)}{\left(\frac{1}{7}\right)(0{\cdot}2) + \left(\frac{2}{7}\right)(0{\cdot}5) + \left(\frac{4}{7}\right)(0{\cdot}7)}$$

$$= \frac{2{\cdot}8}{0{\cdot}2 + 1 + 2{\cdot}8} = \frac{2{\cdot}8}{4} = 0{\cdot}7,$$

which is the required probability.

SECTION—E

(This section comparies of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36. Sherlin and Danju are playing Ludo at home during Covid-19. While rolling the dice, Sherlin's sister Raji observed and noted the possible outcomes of the throw every time belongs to set {1, 2, 3, 4, 5, 6}. Let A be the set of players while B be the set of all possible outcomes.

A = {S, D}, B = {1, 2, 3, 4, 5, 6}

Based on the above information, answer the following :

(I) Prove that $R : B \to B$ defined by
$R = \{(x, y) : y \text{ is divisible by } x\}$ is:
reflexive and transitive but not symmetric.

(II). Raji wants to know the number of functions from A to B. How many number of functions are possible ?

(III) Raji wants to know the number of relations possible from A to B. How many number of relations are possible ?

OR

Let $R : B \to B$ be defined by R={(1, 1), (1, 2), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}, then R is:
reflexive and symmetric. Is it tue ?

Solutions

(I) $x\,R\,x \Rightarrow x$ is divisible by $x \Rightarrow$ R is reflexive
$x\,R\,y \Rightarrow y$ is divisible x but x may not be divisible by $y \Rightarrow$ R is not symmetric
xRy and $yRz \Rightarrow$ y is divisible by x and z is divisible by y
$\Rightarrow$ z is divisible by x
$\Rightarrow$ R is transitive
Hence R is reflexive, transitive and not symmetric.

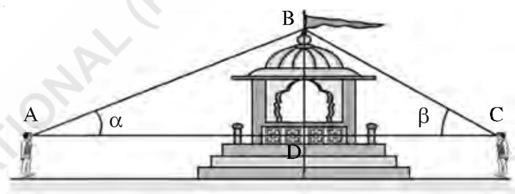
(II) Number of function $\leq 6^2$.

(III) Number of possible relation $= 2^{2 \times 6} = 2^{12}$.

OR

Clearly R is reflexive
[$\because$ *(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) all* $\in$ *R*]
R is not symmetric [$\because$ *(1, 2)* $\in$ *R but (2, 1)* $\notin$ *R*)]
R is transitive
[$\because$ *(1, 1), (1, 2)* $\in$ *R* $\Rightarrow$ *(1, 2)* $\in$ *R; etc.*)]
Then R is reflexive and transitive.

Q. 37. Two men on either side of a temple of 30 meters high observe its top at the angles of elevation α and β respectively (as shown in the figure below). The distance between the two men is $40\sqrt{3}$ meters and the distance between the first person A and the temple is $30\sqrt{3}$ meters.



Based on the above information, answer the following:

(I) Find α, ∠CAB.

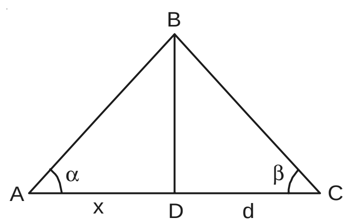
(II) Find β, ∠BCA.

(III) Find whether $\angle ABC = \frac{\pi}{6}$?

OR

Find domain and range of $\cos^{-1} x$.

Solutions



We have

$DB = 30$ m, $AC = 40\sqrt{3}$ m,

$AD = 30\sqrt{3}$ m ,

$DC = 40\sqrt{3} - 30\sqrt{3} = 10\sqrt{3}$

$\therefore$ $AB = \sqrt{(30\sqrt{3})^2 + (30)^2}$

$= \sqrt{2700 + 900} = \sqrt{3600} = 60$ m.

(I) $\sin \alpha = \frac{BD}{AB} = \frac{30}{60} = \frac{1}{2}$

$\therefore$ $\alpha = \sin^{-1}\left(\frac{1}{2}\right)$

(II) $\tan \beta = \frac{BD}{BC} = \frac{30}{10\sqrt{3}} = \sqrt{3}$

$\therefore$ $\beta = \tan^{-1}(\sqrt{3})$.

(III) From (*i*), $\alpha = 30°$

From (*ii*), $\beta = 60°$

$\therefore$ $\alpha + \beta = 90°$

$\therefore$ $\angle ABC = 180° - (90°) = 90° = \frac{\pi}{2}$

No, $\angle ABC \neq \frac{\pi}{6}$.

OR

Domain and Range of $\cos^{-1} x$ are $[-1, 1]$ and $[0, \pi]$ respectively.

Q. 38. The reliability of a COVID PCR test is specified as follows :

Of people having COVID, 90% of the test detects the disease by 10% goes undetected. Of people free of COVID, 99% of the test is judged COVID negative but 1% are diagnosed as showing COVID positive. From a large population of which only 0.1% have COVID, one person is selected at random, given the COVID PCR test and the pathologist reports him/her as COVID positive.

Based on the above information, answer the following :

(I) What is the probability of the 'person to be tested as COVID positive' given that 'he is actually having COVID' ?

(II) What is the probability of the 'person to be tested as COVID positive' given that 'he is actually not having COVID' ?

(III) What is the probability that the 'person is actually having COVID' given that 'he is tested as COVID positive' ?

Or

What is the probability that the 'person selected will be diagnosed as COVID positive' ?

Ans.

(I) 0.9 (II) 0.01 (III) 0.083 ***Or*** 0.01089.

Holy Faith New Style Sample Paper–2

(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions : Same as Holy Faith New Style Sample Paper–1.

SECTION—A

(Multiple choice questions, each question carries 1 mark)

Q. 1. Let R be a relation in the set N given by :

R = {(*a*, *b*) : *a* = *b* – 2, *b* > 6}. Then :

(*a*) (8, 7) ∈ R (*b*) (6, 8) ∈ R

(*c*) (3, 8) ∈ R (*d*) (2, 4) ∈ R.

Ans. (*b*) (6, 8) ∈ R

Q. 2. The principal value of $\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is :

(*a*) $\frac{\pi}{12}$ (*b*) π

(*c*) $\frac{\pi}{3}$ (*d*) $\frac{\pi}{6}$.

Ans. (*a*) $\frac{\pi}{12}$

Q. 3. If for a square matrix A, $A^2 - A + I = O$, then A^{-1} equals :

(*a*) A (*b*) A + I

(*c*) I – A (*d*) A – I.

Ans. (*c*) I – A

Q. 4. For two matrices

$$P = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } Q^T = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \text{ P – Q is :}$$

(*a*) $\begin{bmatrix} 2 & 3 \\ -3 & 0 \\ 0 & -3 \end{bmatrix}$ (*b*) $\begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$

(*c*) $\begin{bmatrix} 4 & 3 \\ 0 & -3 \\ -1 & -2 \end{bmatrix}$ (*d*) $\begin{bmatrix} 2 & 3 \\ 0 & -3 \\ 0 & -3 \end{bmatrix}$.

Ans. (*b*) $\begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$

Q. 5. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $A^{-1} = kA$, then '*k*' is equal to :

(*a*) 19 (*b*) $\frac{1}{19}$

(*c*) $-\frac{1}{19}$ (*d*) – 19.

Ans. (*b*) $\frac{1}{19}$

Q. 6. The set of all points where the function *f* (*x*) = *x* + | *x* | is differentiable, is :

(*a*) (0, ∞) (*b*) (–∞, 0)

(*c*) (–∞, 0) ∪ (0, ∞) (*d*) (–∞, ∞).

Ans. (*c*) (–∞, 0) ∪ (0, ∞)

Q. 7. If *x* = A cos 4*t* + B sin 4*t*, then $\frac{d^2x}{dt^2}$ is equal to :

(*a*) *x* (*b*) –*x*

(*c*) 16*x* (*d*) –16*x*.

Ans. (*d*) –16*x*.

Q. 8. The function $f(x) = x^x$ has a stationary point at

(*a*) $x = e$ (*b*) $x = \frac{1}{e}$

(*c*) $x = 1$ (*d*) $x = \sqrt{e}$.

Ans. (*b*) $x = \frac{1}{e}$

Q. 9. $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\, dx$ **equals :**

(*a*) $10^x - x^{10} + c$ (*b*) $10^x + x^{10} + c$

(*c*) $(10^x - x^{10})^{-1} + c$ (*d*) $\log(10^x + x^{10}) + c$.

Ans. (*d*) $\log(10^x + x^{10}) + c$.

Q. 10. The value of $\int_{-\pi/2}^{\pi/2} x^3 \sin^4 x\, dx$ **is :**

(*a*) 0 (*b*) $\frac{\pi}{2}$

(*c*) π (*d*) $\frac{\pi}{2}$.

Ans. (*a*) 0

Q. 11. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is :

(*a*) π (*b*) $\frac{\pi}{2}$

(*c*) $\frac{\pi}{3}$ (*d*) $\frac{\pi}{4}$.

Ans. (*a*) π

Q. 12. The general solution of the differential equation : $ydx - xdy = 0$; (Given $x, y > 0$) is of the form :

(*a*) $xy = c$ (*b*) $x = cy^2$

(*c*) $y = cx$ (*d*) $y = cx^2$.

Ans. (*c*) $y = cx$

Q. 13.The sum of order and degree of the differential equation :

$$\frac{d^2y}{dx^2} = \left(1 + \frac{dy}{dx}\right)^5 \text{ is :}$$

(*a*) 3 (*b*) 4

(*c*) 5 (*d*) 8.

Ans. (*b*) 4

Q. 14. In Δ ABC, which of the following is not true ?

(*a*) $\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$

(*b*) $\vec{AB} + \vec{BC} - \vec{AC} = \vec{0}$

(*c*) $\vec{AB} + \vec{BC} - \vec{CA} = \vec{0}$

(*d*) $\vec{AB} - \vec{CB} + \vec{CA} = \vec{0}$.

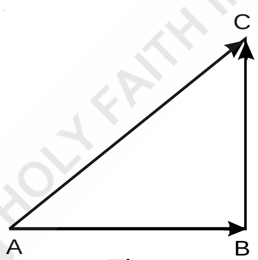


Fig.

Ans. (*c*) $\vec{AB} + \vec{BC} - \vec{CA} = \vec{0}$

Q. 15. The value of λ for which two vectors : $2\hat{i} - \hat{j} + 2\hat{k}$ and $3\hat{i} + \lambda\hat{j} + \hat{k}$ are perpendicular is :

(*a*) 2 (*b*) 4

(*c*) 6 (*d*) 8.

Ans. (*d*) 8.

Q. 16. A line *m* passes through the point (– 4, 2, – 3) and is parallel to line *n*, given by :

$$\frac{-x-2}{4} = \frac{y+3}{-2} = \frac{2z-6}{3}.$$

The vector equation of line *m* is given by :

$$\vec{r} = (-4\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda(p\hat{i} + q\hat{j} + r\hat{k}),$$
where $\lambda \in$ R.

Which of the following could be the possible values for *p*, *q* and *r* ?

(*a*) $p = 4, q = (-2), r = 3$

(*b*) $p = (-4), q = (-2), r = 3$

(*c*) $p = (-2), q = 3, r = (-6)$

(*d*) $p = 8, q = 4, r = (-3)$.

Ans. (*a*) $p = 4, q = (-2), r = 3$

Q. 17. The corner points of the feasible region for the linear programing problem are (0, 2), (3, 0), (6, 0), (6, 8) and (0, 5). Let the objective function be Z = 4*x* + 6*y*, then the minimum value of the objective occurs at :

(*a*) (0, 2) only

(*b*) (3, 0) only

(*c*) The mid-point on the line segment joining the points (0, 2) and (3, 0)

(*d*) Any point on the line segment joining the points (0, 2) and (3, 0).

Ans. (*d*)Any point on the line segment joining the points (0, 2) and (3, 0).

Q. 18. Five fair coins are tossed simultaneously. The probability of the events that atleast one head comes up is :

(*a*) $\frac{27}{32}$ (*b*) $\frac{5}{32}$

(*c*) $\frac{31}{32}$ (*d*) $\frac{1}{12}$.

Ans. (*c*) $\frac{31}{32}$

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) Both ‘A’ and ‘R’ are true and ‘R’ is the correct explanation of ‘A’.

(*b*) Both ‘A’ and ‘R’ are true but ‘R’ is not the correct explanation of ‘A’.

(*c*) ‘A’ is true but ‘R’ is false.

(*d*) ‘A’ is false but ‘R’ is true.

Q. 19. Assertion (A) : The differential coefficient of $\sec(\tan^{-1} x)$ with respect to x is $\frac{x}{\sqrt{1+x^2}}$.

Reason (R) : The differential coefficent of the function with respect to x is the first order derivative of the function.

Ans. (*a*) $\frac{d}{dx}[\sec(\tan^{-1} x)]$

$$= \sec(\tan^{-1} x)\tan(\tan^{-1} x).\frac{d}{dx}(\tan^{-1} x)$$

$$= \sec(\sec^{-1}\sqrt{1+x^2}).\,x.\frac{1}{1+x^2}$$

$$= \sqrt{1+x^2}\,.\,x\,.\frac{1}{1+x^2} = \frac{x}{\sqrt{1+x^2}}.$$

Thus (R) is true.
Also, (R) is true.
Hence, (A) and (R) are both true and (R) explains (A).

Q. 20. Assertion (A) : Equation of a line passing through the points (1, 2, 3) and (3, –1, 3) is :

$$\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-3}{0}.$$

Reason (R) : Equation of a line passing through points (x_1, y_1, z_1), (x_2, y_2, z_2) is given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-y_1}.$$

Ans. (*d*) Equation of the line is :

$$\frac{x-3}{1-3} = \frac{y+1}{2+1} = \frac{z-3}{3-3}$$

$$\Rightarrow \quad \frac{x-3}{-2} = \frac{y+1}{3} = \frac{z-3}{0}.$$

Thus (A) is false.
But (R) is true.

Or

Find the principal value of $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Express $\sin^{-1} \frac{\sin x + \cos x}{\sqrt{2}}$,

when $-\frac{\pi}{4} < x < \frac{\pi}{4}$ in the simplest form.

Ans. $\sin^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$ if $-\frac{\pi}{4} < x < \frac{\pi}{4}$

$= \sin^{-1}\left(\sin x\cdot\frac{1}{\sqrt{2}}+\cos x\cdot\frac{1}{\sqrt{2}}\right)$ if $-\frac{\pi}{4} < x < \frac{\pi}{4}$

$= \sin^{-1}\left(\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}\right)$

if $-\frac{\pi}{4} + \frac{\pi}{4} < x + \frac{\pi}{4} < \frac{\pi}{4} + \frac{\pi}{4}$

$= \sin^{-1}\left(\sin\left(x+\frac{\pi}{4}\right)\right)$ if $0 < \left(x+\frac{\pi}{4}\right) < \frac{\pi}{2}$

$= x + \frac{\pi}{4}$.

Or

Find the principal value of $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Solution. Let $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) = y$ so that

$$\cot y = \frac{-1}{\sqrt{3}} = -\cot\frac{\pi}{3} = \cot\left(\pi-\frac{\pi}{3}\right) = \cot\frac{2\pi}{3}.$$

Since range of principal branch of $\cot^{-1}$ is $(0, \pi)$ and $\cot\frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$,

$\therefore$ Principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is $\frac{2\pi}{3}$.

Q. 22. Show that function :

$$\mathbf{f(x) = 4x^3 - 18x^2 + 27x - 7}$$

is always increasing in R.

Ans. We have : $f(x) = 4x^3 - 18x^2 + 27x - 7$.

$$\therefore \quad f'(x) = 12x^2 - 36x + 27$$
$$= 12(x^2 - 3x) + 27$$
$$= 12\left(x^2-3x+\frac{9}{4}\right)+27-27$$
$$= 12\left(x-\frac{3}{2}\right)^2 \geq 0 \ \forall\, x \in \mathbf{R}.$$

Hence, $f(x)$ is always increasing in **R**.

Q. 23. Find the points of local maxima and local minima, if any, of the following function :

$f(x) = \sin x + \frac{1}{2}\cos 2x;\ 0 < x < \frac{\pi}{2}$.

Solution. We have : $f(x) = \sin x + \frac{1}{2}\cos 2x$.

$$\therefore \quad f'(x) = \cos x + \frac{1}{2}(-\sin 2x)(2)$$
$$= \cos x - \sin 2x.$$

For local max./min., $f'(x) = 0$

$\Rightarrow \quad \cos x - \sin 2x = 0$

$\Rightarrow \quad \cos x - 2\sin x\cos x = 0$

$\Rightarrow \quad \cos x\,(1 - 2\sin x) = 0$

$\Rightarrow \quad \cos x = 0$ or $1 - 2\sin x = 0$

$\Rightarrow \cos x = 0$ or $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$.

Thus $\quad x = \frac{\pi}{6}$. $\quad \left[\because \frac{\pi}{6} \in \left(0, \frac{\pi}{2}\right)\right]$

Now $\quad f''(x) = -\sin x - 2\cos 2x.$

$$\therefore \quad f''\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} - 2\cos\frac{2\pi}{6}$$

$$= -\sin\frac{\pi}{6} - 2\cos\frac{\pi}{3} = -\frac{1}{2} - 2.\frac{1}{2}$$

$$= -\frac{1}{2} - 1 = -\frac{3}{2} < 0.$$

Hence, $f(x)$ is local max. at $x = \frac{\pi}{6}$.

Or

A manufacturer can sell 'x' items at a price of ₹ $\left(5 - \frac{x}{100}\right)$ each. The cost price of 'x' items is ₹ $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit. ***(N.C.E.R.T.)***

Solution. Here C (x), cost price = ₹ $\left(\frac{x}{5} + 500\right)$

and S(x), selling price = $\left(5 - \frac{x}{100}\right)x$ = ₹$\left(5x - \frac{x^2}{100}\right)$.

$\therefore$ P(x), profit = S(x) – C (x)

$$= \left(5x - \frac{x^2}{100}\right) - \left(\frac{x}{5} + 500\right)$$

$$= \frac{24}{5}x - \frac{x^2}{100} - 500.$$

$\therefore \quad P'(x) = \frac{24}{5} - \frac{x}{50}$ and $P''(x) = -\frac{1}{50}$.

Now $\quad P'(x) = 0 \Rightarrow \frac{24}{5} - \frac{x}{50} = 0 \Rightarrow x = 240.$

And $P''(240) = -\frac{1}{50} < 0$.

Thus $x = 240$ is the point of maxima.

Hence, the manufacturer should sell 240 items in order to earn maximum profit.

Q. 24. Evaluate : $\displaystyle\int_{-2}^{2} \frac{\mathbf{x^2}}{\mathbf{1+5^x}}\,\mathbf{dx.}$

Solution. Let $\displaystyle I = \int_{-2}^{2} \frac{x^2}{1+5^x}\,dx \qquad ...(1)$

$$\therefore \qquad I = \int_{-2}^{2} \frac{(-2+2-x)^2}{1+5^{(-2+2-x)}}\,dx \quad [\textit{Property III}]$$

$$= \int_{-2}^{2} \frac{(-x)^2}{1+5^{-x}}\,dx$$

$$= \int_{-2}^{2} \frac{5^x x^2}{1+5^x}\,dx \qquad ...(2)$$

Adding (1) and (2), $\displaystyle 2I = \int_{-2}^{2} \frac{1+5^x}{1+5^x}x^2\,dx$

$$= \int_{-2}^{2} x^2 dx = \left[\frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{1}{3}\left[2^3 - (-2)^3\right]$$

$$= \frac{1}{3}[8-(-8)] = \frac{1}{3}[8+8]$$

$$= \frac{16}{3}.$$

Hence, $\quad I = \frac{8}{3}$.

Q. 25. Prove that the greatest integer function [x] is continuous at all points except at integral points.

Solution. Let $f(x) = [x]$.

Case I. When c $\in$ I.

Here $f(x) = \begin{cases} c-1 & \text{if } c-1 \le x < c \\ c & \text{if } \quad c \le x < c+1. \end{cases}$

Now $\displaystyle\lim_{x \to c^-} f(x) = \lim_{x \to c^-} [x]$

$$= \lim_{h \to 0} [c-h] = \lim_{h \to 0} (c-1)$$

$$= c - 1$$

and $\displaystyle\lim_{x \to c^+} f(x) = \lim_{x \to c^+} [x]$

$$= \lim_{h \to 0} [c+h] = \lim_{h \to 0} (c) = c.$$

Thus $\displaystyle\lim_{x \to c^-} f(x) \ne \lim_{x \to c^+} f(x) \qquad [\because c-1 \ne c]$

$\Rightarrow$ 'f' is not continuous at $x = c$.

But 'c' is arbitrary.

Hence, 'f' is not continuous at all integral points.

Case II. When a $\notin$ I.

Here a is any real number. Then there exists an integer c such that $c - 1 < a < c$.

Now $\displaystyle\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} [a-h]$

$$= \lim_{h \to 0} (c-1) = c-1$$

$[\because c-1 < a-h < c \therefore [a-h] = c-1]$

and $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} [a+h]$

$$= \lim_{h \to 0} (c-1) = c-1.$$

[$\because$ *c* – *1*< *a* + *h* < *c* $\therefore$ [*a* + *h*] = *c* – *1*]

Also, $f(a) = c - 1$.

[$\because$ *c* – *1* < *a* < *c* + *a* $\therefore$ [*a*] = *c* – *1*]

Thus $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

[$\because$ *each* = *c* – *1*]

$\Rightarrow$ '*f*' is continuous at $x = a$.

But '*a*' is arbitrary.

Hence, '*f*' is continuous at all real points except integral points.

SECTION—C

(This section comparies of short answer type questions (SA) of 3 marks each)

Q. 26. Evaluate : $\displaystyle\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$.

Solution. Let $\displaystyle I = \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$.

Dividing Num and Denom. by $\cos^2 x$, we get :

$$I = \int \frac{\sec^2 x}{a^2 \tan^2 x + b^2}\, dx.$$

Put tan x = t so that $\sec^2 x\, dx = dt$.

$$\therefore \quad I = \frac{dt}{a^2t^2 + b^2} = \frac{1}{a^2} \frac{dt}{\frac{b^2}{a^2} + t^2}$$

$$= \frac{1}{a^2} \int \frac{dt}{\left(\frac{b}{a}\right)^2 + t^2} \qquad \Big| \text{"Form:} \int \frac{dx}{a^2 + x^2}\text{"}$$

$$= \frac{1}{a^2} \cdot \frac{1}{\frac{b}{a}} \tan^{-1} \frac{t}{\frac{b}{a}} + c$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{a}{b} \tan x\right) + c.$$

Q. 27. A and B throw a pair of dice alternately. A wins the game if he gets a total of 7 and B wins if he gets a total of 10. If A starts the game, find the probability that B wins.

Solution. Number of outcomes = 6 × 6 = 36.
Favourable outcomes for winning of A are :
{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}.

$$\therefore \quad P(\text{A wins}) = P(A) = \frac{6}{36} = \frac{1}{6}$$

and $\quad P(\text{A loses}) = P(\bar{A}) = 1 - \frac{1}{6} = \frac{5}{6}$.

Favourable outcomes for winning of B are :
{(4, 6), (6, 4), (5, 5)}

$$\therefore \quad P(\text{B wins}) = P(B) = \frac{3}{36} = \frac{1}{12}$$

and $\quad P(\text{B loses}) = P(\bar{B}) = 1 - \frac{1}{12} = \frac{11}{12}$.

$$P(\text{B wins}) = P(\bar{A})\, P(B) + P(\bar{A})\, P(\bar{B})\, P(\bar{A})\, P(B) + P(\bar{A})\, P(\bar{B})\, P(\bar{A})\, P(\bar{B})\, P(\bar{A})\, P(B) + \ldots$$

$$= \frac{5}{6} \times \frac{1}{12} + \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{1}{12} + \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{1}{12} \qquad |\,\textbf{G.P.}$$

$$= \frac{\frac{5}{72}}{1 - \frac{5}{6} \times \frac{11}{12}} = \frac{5}{72 - 55} = \frac{5}{17}.$$

Q. 28. Evaluate : $\displaystyle\int \frac{x+2}{\sqrt{x^2 + 5x + 6}}\, dx$.

Solution. $\displaystyle I = \int \frac{x+2}{\sqrt{x^2 + 5x + 6}}\, dx$

$$= \int \frac{\frac{1}{2}(2x+5) + \left(2 - \frac{5}{2}\right)}{\sqrt{x^2 + 5x + 6}}\, dx$$

$$= \frac{1}{2} \int \frac{2x+5}{\sqrt{x^2 + 5x + 6}}\, dx - \frac{1}{2} \int \frac{1}{\sqrt{x^2 + 5x + 6}}\, dx$$

$$= \frac{1}{2} \int (x^2 + 5x + 6)^{-1/2} (2x+5)\, dx$$

$$- \frac{1}{2} \int \frac{1}{\sqrt{\left(x^2 + 5x + \frac{25}{4}\right) + \left(6 - \frac{25}{4}\right)}}\, dx$$

$$= \frac{1}{2} \frac{(x^2 + 5x + 6)^{1/2}}{1/2} - \frac{1}{2} \int \frac{dx}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \frac{1}{4}}}\, dx$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \frac{1}{4}}}dx \qquad ...(1)$$

Let $\quad I_1 = \int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \frac{1}{4}}}.$

Put $x + \frac{5}{2} = t$ so that $dx = dt$.

$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{2}\right)^2}}$

| **"Form:** $\int \frac{dx}{\sqrt{x^2-a^2}}$"

$$= \log\left| t + \sqrt{t^2 - \left(\frac{1}{2}\right)^2}\right|$$

$$= \log\left| \left(x+\frac{5}{2}\right) + \sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right|$$

$$= \log\left| \left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right|.$$

From (1),

$$I = \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + c.$$

Or

Evaluate : $\displaystyle\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x}dx.$ ***(C.B.S.E. 2018)***

Solution. Let $\quad I = \displaystyle\int_0^{\pi/4} \frac{\sin x + \cos x}{16+9\sin 2x}dx$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9[1-(\sin x - \cos x)^2]}dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 9(\sin x - \cos x)^2}dx.$$

Put $\sin x - \cos x = t$ so that $(\cos x + \sin x)\, dx = dt$.

When $x = 0$, $\quad t = 0 - 1 = -1$.

When $x = \frac{\pi}{4}$, $\quad t = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0.$

$\therefore \quad I = \displaystyle\int_{-1}^{0} \frac{1}{25 - 9t^2}dt$

$$= \frac{1}{9}\int_{-1}^{0} \frac{1}{\left(\frac{5}{3}\right)^2 - t^2}dt$$

| **"Form :** $\int \frac{1}{a^2 - x^2}dx$"

$$= \frac{1}{9}\left[\frac{1}{2\left(\frac{5}{3}\right)}\log\left|\frac{\frac{5}{3}+t}{\frac{5}{3}-t}\right|\right]_{-1}^{0}$$

$$= \frac{1}{9} \times \frac{1}{\frac{10}{3}}\left[\log\left|\frac{5/3}{5/3}\right| - \log\left|\frac{2/3}{8/3}\right|\right]$$

$$= \frac{1}{30}\left[\log|1| - \log\left|\frac{1}{4}\right|\right]$$

$$= \frac{1}{30}\left[\log 1 - \log\frac{1}{4}\right] = \frac{1}{30}\log 4$$

$$= \frac{1}{15}\log 2.$$

Example 29. Solve :

$$\left(\mathbf{x \cos \frac{y}{x}}\right) \mathbf{(y\, dx + x\, dy)} = \left(\mathbf{y \sin \frac{y}{x}}\right) \mathbf{(x\, dy - y\, dx).}$$

Solution. The given equation is :

$$\left(x\cos\frac{y}{x}\right)(y\,dx + x\,dy) = \left(y\sin\frac{y}{x}\right)(x\,dy - y\,dx)$$

$$\Rightarrow \left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\,.\,\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left\{x\cos\frac{y}{x} + y\sin\frac{y}{x}\right\}y}{\left\{y\sin\frac{y}{x} - x\cos\frac{y}{x}\right\}x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left[\cos\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)\right]\left(\frac{y}{x}\right)}{\left[\left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)\right]}$$

[*Dividing num. and denom. by* x^2]

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get :

$$v + x\frac{dv}{dx} = \frac{(\cos v + v\sin v)\,v}{v\sin v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{v\sin v - \cos v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad \frac{v\sin v - \cos v}{v\cos v}\,dv = 2\frac{dx}{x}. \qquad |\textit{Variables Separable}$$

Integrating, $\int \frac{v\sin v - \cos v}{v\cos v}\,dv = 2\int \frac{dx}{x} + c'$

$$\Rightarrow \quad \int \frac{\sin v}{\cos v}\,dv - \int \frac{dv}{v} = 2\int \frac{dx}{x} + c'$$

$$\Rightarrow \quad -\log|\cos v| - \log|v| = 2\log|x| - \log|c|,$$

where $c' = -\log|c|$

$$\Rightarrow \quad \log|x^2| + \log|\cos v| + \log|v| = \log|c|$$

$$\Rightarrow \quad \log|x^2 v\cos v| = \log|c|$$

$$\Rightarrow \quad x^2 v\cos v = c$$

$$\Rightarrow \quad x^2\frac{y}{x}\cos\frac{y}{x} = c \quad \Rightarrow \quad xy\cos\frac{y}{x} = c,$$

which is the the reqd. solution.

Or

(*i*) Solve the differential equation :

$(\tan^{-1} y - x)\,dy = (1 + y^2)\,dx$.

(*N.C.E.R.T.; C.B.S.E. 2015*)

(*ii*) Find the particular solution when x = 0, y = 0.

(*A.I. C.B.S.E. 2013*)

Solution. (*i*) The given equation can be written as :

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \quad ...(1) \;|\; \textit{Linear Equation}$$

Comparing with $\frac{dx}{dy} + Px = Q$, we have :

'P' $= \frac{1}{1+y^2}$ and 'Q' $= \frac{\tan^{-1} y}{1+y^2}$.

$$\therefore \quad \text{I.F.} = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}.$$

Multiplying (1) by $e^{\tan^{-1} y}$, we get :

$$e^{\tan^{-1} y}.\frac{dx}{dy} + \frac{x}{1+y^2}e^{\tan^{-1} y} = \frac{\tan^{-1} y}{1+y^2}.e^{\tan^{-1} y}$$

$$\Rightarrow \quad \frac{d}{dy}\left(x.e^{\tan^{-1} y}\right) = \left(\frac{\tan^{-1} y}{1+y^2}\right)e^{\tan^{-1} y}.$$

Integrating, $x\,e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1+y^2}\right)e^{\tan^{-1} y}dy + c$ ...(2)

Now $\quad I = \int \left(\frac{\tan^{-1} y}{1+y^2}\right)e^{\tan^{-1} y}dy.$

Put $\tan^{-1} y = t$ so that $\left(\frac{1}{1+y^2}\right)dy = dt.$

$$\therefore \quad I = \int t\,e^t\,dt = t\,e^t - \int (1)\,e^t\,dt$$

[*Integrating by parts*]

$$= te^t - e^t = e^t(t-1)$$

$$= e^{\tan^{-1} y}(\tan^{-1} y - 1).$$

$\therefore$ From (2), $x\,e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$

$$\Rightarrow \quad x = (\tan^{-1} y - 1) + c\,e^{-\tan^{-1} y} \quad ...(3),$$

which is the reqd. solution.

(*ii*) When $x = 0, y = 0.$

$$\therefore \quad 0 = (\tan^{-1} 0 - 1) + ce^{-\tan^{-1} 0}$$

$$\Rightarrow \quad 0 = (0-1) + ce^{-0}$$

$$\Rightarrow \quad 0 = -1 + c \Rightarrow c = 1.$$

Putting in (3), $x = (\tan^{-1} y - 1) + e^{-\tan^{-1} y}$,

which is the reqd. particular solution.

Q. 30. Maximise and Minimise :

$Z = 4x + 3y - 7$

subject to the constraints :

$x + y \le 10, x + y \ge 3, x \le 8, y \le 9, x, y \ge 0.$

Solution. The given system of constraints is :

$x + y \le 10$ ...(1)

$x + y \ge 3$ ...(2)

$x \le 8$ ...(3)

$y \le 9$ ...(4)

$x, y \ge 0$ ...(5)

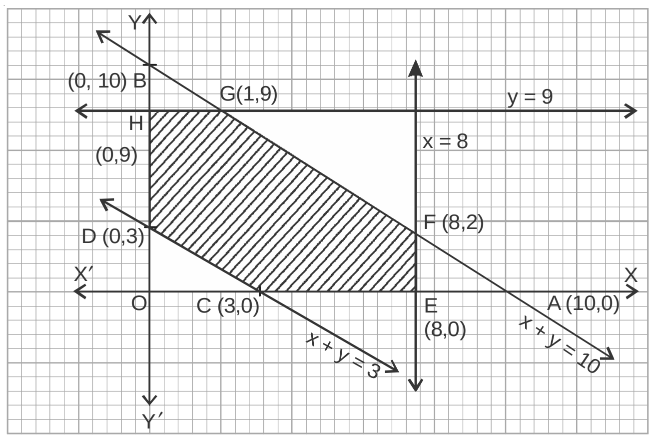


Fig.

The shaded region in the above figure is the feasible region determined by the system of constraints (1) – (5). It is observed that the feasible region DCEFGH is bounded. Thus we use ***Corner Point Method*** to determine the maximum and minimum values of Z, where :

$$Z = 4x + 3y - 7 \quad ...(6)$$

The co-ordinates of D, C, E, F, G and H are respectively (0, 3), (3, 0), (8, 0), (8, 2),

[*Solving x = 8 and x + y = 10*]

(1, 9) [*Solving y = 9 and x + y = 10*]

and (0, 9).

We evaluate Z at each corner point:

Corner Point	Corresponding Value of Z
D : (0, 3)	**2 (Minimum)**
C : (3, 0)	5
E : (8, 0)	25
F : (8, 2)	**31 (Maximum)**
G : (1, 9)	24
H : (0, 9)	20

Hence, $Z_{max} = 31$ at (8, 2) and $Z_{min} = 2$ at (0, 3).

Or

Minimize and Maximize Z = 5x + 2y, subject to the following constraints :

x – 2y ≤ 2, 3x + 2y ≤ 12, –3x + 2y ≤ 3, x ≥ 0, y ≥ 0.

(*A.I.C.B.S.E. 2015*)

Solution. The given system of constraints is :

$$x - 2y \le 2 \quad ...(1)$$
$$3x + 2y \le 12 \quad ...(2)$$
$$-3x + 2y \le 3 \quad ...(3)$$
and $$x \ge 0, y \ge 0 \quad ...(4)$$

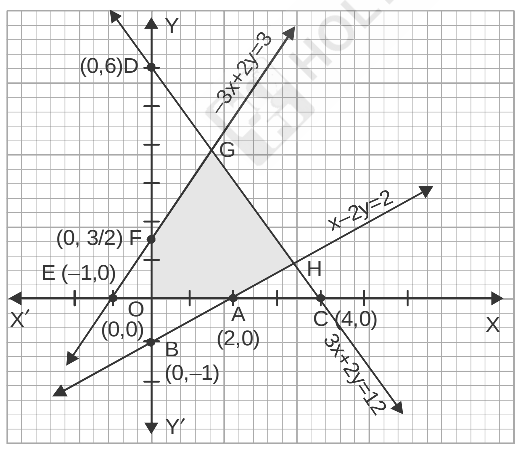


Fig.

The shaded region in the above figure is the feasible region determined by the system of constraints (1) – (4). It is observed that the feasible region OAHGF is bounded. Thus we use ***Corner Point Method*** to determine the maximum and minimum value of Z, where

$$Z = 5x + 2y \quad ...(5)$$

The co-ordinates of O, A, H, G and F are :

(0, 0), (2, 0), $\left(\frac{7}{2},\frac{3}{4}\right)$, $\left(\frac{3}{2},\frac{15}{4}\right)$ and $\left(0,\frac{3}{2}\right)$ respectively.

[*Solving x – 2y = 2 and 3x + 2y = 12 for H and –3x + 2y = 3 and 3x + 2y = 12 for G*]

We evaluate Z at each corrner point :

Corner Point	Corresponding value of Z
O : (0, 0)	**0 (Minimum)**
A : (2, 0)	10
H : $\left(\frac{7}{2},\frac{3}{4}\right)$	**19 (Maximum)**
G : $\left(\frac{3}{2},\frac{15}{4}\right)$	15
F : $\left(0,\frac{3}{2}\right)$	3

Hence, $Z_{max} = 19$ at $\left(\frac{7}{2},\frac{3}{4}\right)$ and $Z_{min} = 0$ at (0, 0).

Q. 31. Minimize and Maximize Z = 5x + 2y, subject to the following constraints :

x – 2y ≤ 2, 3x + 2y ≤ 12, –3x + 2y ≤ 3, x ≥ 0, y ≥ 0.

(*A.I.C.B.S.E. 2015*)

Solution. The given system of constraints is :

$$x - 2y \le 2 \quad ...(1)$$
$$3x + 2y \le 12 \quad ...(2)$$
$$-3x + 2y \le 3 \quad ...(3)$$
and $$x \ge 0, y \ge 0 \quad ...(4)$$

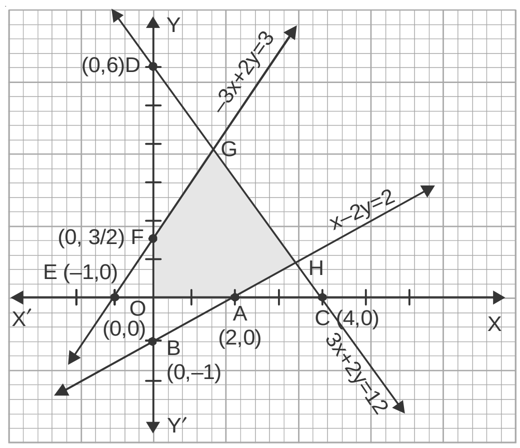


Fig.

The shaded region in the above figure is the feasible region determined by the system of constraints (1) – (4). It is observed that the feasible region OAHGF is bounded. Thus we use ***Corner Point Method*** to determine the maximum and minimum value of Z, where

$$Z = 5x + 2y \quad ...(5)$$

The co-ordinates of O, A, H, G and F are :

(0, 0), (2, 0), $\left(\frac{7}{2},\frac{3}{4}\right)$, $\left(\frac{3}{2},\frac{15}{4}\right)$ and $\left(0,\frac{3}{2}\right)$ respectively.

[*Solving x – 2y = 2 and 3x + 2y = 12 for H and –3x + 2y = 3 and 3x + 2y = 12 for G*]

We evaluate Z at each corrner point :

Corner Point	Corresponding value of Z
O : (0, 0)	**0 (Minimum)**
A : (2, 0)	10
H : $\left(\frac{7}{2}, \frac{3}{4}\right)$	**19 (Maximum)**
G : $\left(\frac{3}{2}, \frac{15}{4}\right)$	15
F : $\left(0, \frac{3}{2}\right)$	3

Hence, $Z_{max} = 19$ at $\left(\frac{7}{2}, \frac{3}{4}\right)$ and $Z_{min} = 0$ at (0, 0).

SECTION—D

(This section comparies of Long Answer Type questions (LAQ) of 5 marks each)

Q. 32. Find the area of the parabola $y^2 = 4ax$, bounded by the latus-rectum.

Solution. $y^2 = 4ax$ is a parabola with focus S(a, 0).

Here L L′ ($x = a$) is the latus-rectum.

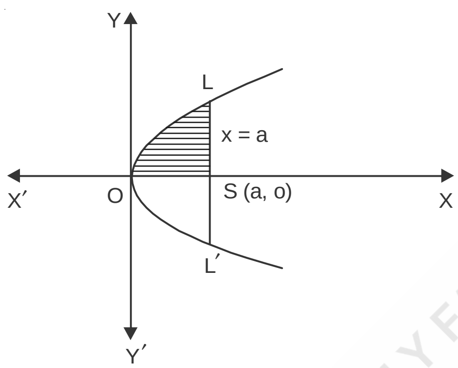


Fig.

$\therefore$ Reqd. area = 2 (area OSL)

$$= 2\int_0^a \sqrt{4ax}\,dx = 4\sqrt{a}\int_0^a x^{1/2}\,dx$$

$$= 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}\left[a^{3/2} - 0\right] = \frac{8}{3}a^2 \text{ sq. units.}$$

Q. 33. Check whether the relation *R* in the set N of natural numbers given by :

***R* = {(*a*, *b*) : *a* is a divisor of *b*}**

is reflexive, symmetric or transitive.

Also, determine whether *R* is an equivalence relation.

Solution. R = {(a, b) : a is a divisor of b}

A = **N**, set of all natural numbers.

(I) R is reflexive.

[$\because$ (a, a) $\in$ R $\forall$ $a \in$ **N** as a is a divisor of a]

(II) R is not symmetric.

[$\because$ *(a, b) ∈ R ⇒ a is a divisor of b but (b, a) ∉ R ⇒ b is not a divisor of a*]

For Ex. *In (3, 6), 3 is a divisor of 6 but 6 is not a divisor of 3*]

(III) R is transitive.

[$\because$ *(a, b) ∈ R, (b, c) ∈ R ⇒ (a, c) ∈ R as a is a divisor of b and b is a divisor of c ⇒ a is a divisor of c*]

Thus, R is reflexive and transitive but not symmetric.

Hence, R is not an equivalence relation.

Or

Show that f : N → N given by :

$$\mathbf{f(x)} = \begin{cases} \mathbf{x+1, \text{ if } x \text{ is odd}} \\ \mathbf{x-1, \text{ if } x \text{ is even}} \end{cases}$$

is both one–one and onto.

Solution. One–One.

Here we discuss the following possible cases :

(I) *When x_1 is odd and x_2 is even.*

Here $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 - 1$

$\Rightarrow x_2 - x_1 = 2$, which is impossible.

(II) *When x_1 is even and x_2 is odd.*

Here $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 + 1$

$\Rightarrow x_1 - x_2 = 2$, which is impossible.

(III) *When x_1 and x_2 are both odd.*

Here $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$.

$\therefore$ 'f' is one–one.

(IV) *When x_1 and x_2 are both even.*

Here $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$.

$\therefore$ 'f' is one–one.

Onto. Let 'x' be an arbitrary natural number.

When x is an odd natural number, then there exists an even natural number (x + 1) such that :

$$f(x + 1) = (x + 1) - 1 = x.$$

When x is an even natural number, then there exists an odd natural number (x – 1) such that :

$$f(x - 1) = (x - 1) + 1 = x.$$

$\therefore$ Each $x \in$ **N** has its pre-image in **N**.

Thus 'f' is onto.

Hence, 'f' is both one–one and onto.

Q. 34. Let $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{i} - \hat{j}$ and $\vec{c} = \hat{i} + \hat{j} + \hat{k}$. If $\hat{n}$ is a unit vector such $\vec{a}.\hat{n} = 0$ and $\vec{b}.\hat{n} = 0$, then find $|\vec{c}.\hat{n}|$.

Solution. Let $\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$,

where $\sqrt{x^2 + y^2 + z^2} = 1$

i.e. $x^2 + y^2 + z^2 = 1$ ...(1)

Now $\vec{a}.\hat{n} = 0$

$\Rightarrow (\hat{i} + \hat{j}).(x\hat{i} + y\hat{j} + z\hat{k}) = 0$

$\Rightarrow x + y = 0$ ....(2)

Again, $\vec{b}.\hat{n} = 0$

$\Rightarrow \quad (\hat{i} - \hat{j})(x\hat{i} + y\hat{j} + z\hat{k}) = 0$

$\Rightarrow \quad x - y = 0 \quad$ ....(3)

Solving (2) and (3), $x = 0 = y$.

Putting in (1), $0 + 0 + z^2 = 1 \Rightarrow z = \pm 1$.

Now $\quad \vec{c}.\hat{n} = (\hat{i} + \hat{j} + \hat{k}).(x\hat{i} + y\hat{j} + z\hat{k})$

$= x + y + z = 0 + 0 \pm 1 = \pm 1.$

Hence, $\quad |\vec{c}.\hat{n}| = |\pm 1| = 1.$

Or

A line makes angles α, β, γ, δ with the four diagonals of a cube, prove that :

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}.$$

Solution. Let O be the origin and OA, OB, OC (each = a) be the axes.

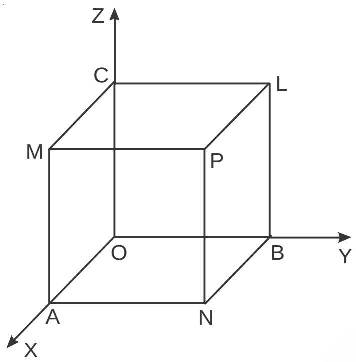


Fig.

Thus the co-ordinates of the points are :

O (0, 0, 0), A (a, 0, 0), B (0, a, 0), C (0, 0, a), P (a, a, a), L (0, a, a), M (a, 0, a), N (a, a, 0).

Here OP, AL, BM and CN are four diagonals.

Let < l, m, n > be the direction-cosines of the given line.

Now direction-ratios of OP are < a – 0, a – 0, a – 0 > *i.e.* < a, a, a > *i.e.* < 1, 1, 1 >,

direction-ratios of AL are :

< 0 – a, a – 0, a – 0 > *i.e.* < – a, a, a > *i.e.* < –1, 1, 1 >,

direction-ratios of BM are :

< a – 0, 0 – a, a – 0 > *i.e.* < a, – a, a > *i.e.* < 1, –1, 1 >

and direction-ratios of CN are :

< a – 0, a – 0, 0 – a > *i.e.* < a, a, – a > *i.e.* < 1, 1, – 1 >.

Thus the direction-cosines of OP are :

$$< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} > ;$$

the direction-cosines of AL are $< \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$;

the direction-cosines of BM are $< \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$

and the direction-cosines of CN are :

$$< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} >.$$

If the given line makes an angle 'α' with OP, then :

$$\cos \alpha = \left| l\left(\frac{1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \right|.$$

$$\therefore \quad \cos \alpha = \frac{|l + m + n|}{\sqrt{3}} \quad ...(1)$$

If the given line makes an angle 'β' with AL, then :

$$\cos \beta = \left| l\left(-\frac{1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \right|$$

$$\therefore \quad \cos \beta = \frac{|-l + m + n|}{\sqrt{3}} \quad ...(2)$$

Similarly, $\cos \gamma = \frac{|l - m + n|}{\sqrt{3}}$ ...(3)

and $\cos \delta = \frac{|l + m - n|}{\sqrt{3}}$ ...(4)

Squaring and adding (1), (2), (3) and (4), we get :

$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta$

$$= \frac{1}{3} [(l + m + n)^2 + (-l + m + n)^2 + (l - m + n)^2 + (l + m - n)^2]$$

$$= \frac{1}{3} [4 (l^2 + m^2 + n^2)] = \frac{1}{3} [4 (1)].$$

Hence, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}$.

Q. 35. A company follows a model of bifurcating the tasks into the catagories shown below :

	URGENT	NOT URGENT
IMPORTANT	Urgent and Important	Not urgent but Important
NOT IMPORTANT	Urgent but not Important	Not urgent and not Important

At the beging of a financial year, it was noticed that :

- **40% of the total tasks were urgent and the rest were not**
- **half of the urgent tasks were imported and**
- **30% of tasks that were not urgent, were not important.**

What a the probability that a randomly selected task that is not important is urgent ? Use Bayes' theorem and show your steps.

Solution. Here, P (urgent) = $\frac{40}{100}$,

P (not urgent) = $\frac{60}{100}$

P (Not important/Urgent) = $\frac{1}{2}$ and

P (Not important/Not urgent) = $\frac{30}{100}$.

By *Bayes' Theorem,*

P (Urgent/Not Important)

$$= \frac{P\text{ (Urgent)} \times P\text{ (Not important/Urgent)}}{P\text{ (Urgent)} \times P\text{ (Not important/Urgent)} + P\text{ (Not Urgent)} + P\text{ (Important/Not Urgent)}}$$

$$= \frac{\frac{40}{100} \times \frac{1}{2}}{\frac{40}{100} \times \frac{1}{2} + \frac{60}{100} \times \frac{30}{100}}$$

$$= \frac{10}{19} \textit{ Or } 52.63\% .$$

SECTION—E

(This section comparies of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36. A manufacturer produces three stationery products Pencil, Eraser and Sharpener which he sells in two markets. Annual sales are indicated below :

Market	Products (in numbers)		
	Pencil	**Eraser**	**Sharpener**
A	**10,000**	**2,000**	**18,000**
B	**6,000**	**20,000**	**8,000**

If the unit sale price of Pencil, Eraser and Sharpener are ₹ 2.50, ₹ 1.50 and ₹ 1.00 respectively, and unit cost of the above three commodities are ₹ 2.00, ₹ 1.00 and ₹ 0.50 respectively, then :

Based on the above information, answer the following :

(I) Find the revenue of market A.

(II) Find the total revenue of market B.

(III) Find the profit in market A and B.

Or

Find the gross profit in both markets.

Solutions

(I) 10,000 × 2.5 + 2,000 × 1.50 + 18,000 × 1

= 25,000 + 3,000 + 18,000 = ₹ 46,000.

(II) 6,000 × 2.5 + 20,000 × 1.5 + 8,000 × 1

= 15,000 + 30,000 + 8,000 = ₹ 53,000.

(III) Profit in market A

10,000 × 0.50 + 2,000 × 0.50 + 18,000 × 0.50 = 5,000 + 1,000 + 9,000 = ₹ 15,000.

Profit in market B

6,000 × 0.50 + 20,000 × 0.50 + 8,000 × 0.50 = 3,000 + 10,000 + 4,000 = ₹ 17,000.

Or

Gross Profit

(10,000 + 6,000) × 0.50 + (2,000 + 20,000) × 0.50 + (18,000 + 8,000) × 0.50 = 8,000 + 11,000 + 13,000 = ₹ 32,000.

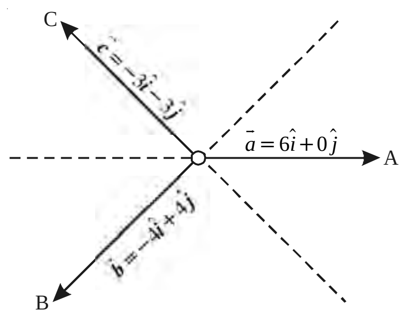
Q. 37. Read the following passage and answer the questions given below:

Teams A, B, C went for playing a tug of war game. Teams A, B, C have attached a rope to a metal ring and is trying to pull the ring into their own area.

Team A pulls with force $F_1 = 6\hat{i} + 0\hat{j}\ kN,$

Team B pulls with force $F_2 = -4\hat{i} + 4\hat{j}\ kN,$

Team C pulls with force $F_3 = -3\hat{i} - 3\hat{j}\ kN,$



(I) What is the magnitude of the force of Team A ?

(II) Which team will win the game?

(III) Find the magnitude of the resultant force exerted by the terms.

Or

In what direction is the ring getting pulled?

Solutions

Here, $|\vec{F}_1| = \sqrt{6^2+0^2} = 6\ k\text{N}$

$|\vec{F}_2| = \sqrt{(-4)^2+4^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}\ k\text{N}$

$|\vec{F}_3| = \sqrt{(-3)^2+(-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}\ k\text{N}$

(I) Magnitude of force of Team A = 6 kN.

(II) Since, 6 kN is greatest,

$\therefore$ team (A) will win the game.

(III) $|\vec{F}_1| = \vec{F}_1+\vec{F}_2+\vec{F}_3$

$= 6\hat{i}+\hat{j}-4\hat{i}+4\hat{j}-3\hat{i}-3\hat{j} = -\hat{i}+\hat{j}$

$\therefore\ |\vec{F}| = \sqrt{(-1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}\ k\text{N}.$

Or

Here, $\vec{F} = -\hat{i}+\hat{j}$

$\therefore\ \theta = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi-\frac{\pi}{4} = \frac{3\pi}{4},$

Where 'θ' is the angle made by the resultant force with the +ve direction of x-axis.

Q. 38. Two motorcycles A and B are running at the speed more than the allowed speed on the roads represented by the lines:

$\vec{r} = \lambda(\hat{i}+2\hat{j}-\hat{k})$

and $\vec{r} = (3\hat{i}+3\hat{j})+\mu(2\hat{i}+\hat{j}+\hat{k})$ respectively.

Based on the above information, answer the following questions:

(a) Find the shortest distance between the given lines

(b) Find the point at which the motorcycles may collide.

Solutions

(*a*) The given lines are:

$\vec{r} = \lambda(\hat{i}+2\hat{j}-\hat{k})$ ...(1)

and $\vec{r} = (3\hat{i}+3\hat{j})+\mu(2\hat{i}+\hat{j}+\hat{k})$ ...(2)

Comparing given equations with:

$\vec{r} = \vec{a}_1+\lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2+\mu\vec{b}_2$, we have :

$\vec{b}_1 = \hat{i}+2\hat{j}-\hat{k},\ \vec{b}_2 = 2\hat{i}+\hat{j}+\hat{k}$

and $\vec{a}_1 = 0\hat{i}+0\hat{j}+0\hat{k},\ \vec{a}_2 = 3\hat{i}+3\hat{j}.$

Now $\vec{b}_1\times\vec{b}_2 = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -1\\ 2 & 1 & 1\end{vmatrix}$

$= \hat{i}(2+1)-\hat{j}(1+2)+\hat{k}(1-4)$

$= 3\hat{i}-3\hat{j}-3\hat{k}.$

$\therefore\ |\vec{b}_1\times\vec{b}_2| = \sqrt{3^2+(-3)^2+(-3)^2}$

$= \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}.$

Also, $\vec{a}_2-\vec{a}_1 = 3\hat{i}+3\hat{j}.$

$\therefore$ Shortest distance (S.D.) $= \left|\frac{(\vec{b}_1\times\vec{b}_2).(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right|$

$= \left|\frac{(3\hat{i}-3\hat{j}-3\hat{k}).(3\hat{i}+3\hat{j})}{3\sqrt{3}}\right|$

$= \frac{(3)(3)+(-3)(3)-0}{3\sqrt{3}} = 0.$

(*b*) The given lines are:

$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ ...(1)

and $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ ...(2)

Any point on (1) is $(k_1, 2k_1, -k_1)$.

Any point on (2) is $(2k_2+3, k_2+3, k_2)$.

The lines meet when $k_1 = 2k_2+3$,

$2k_1 = k_2+3,\ -k_1 = k_2$.

Solving first two, $k_1 = 1$ and $k_2 = -1$.

These also satisfy the third equation.

Hence, the accident occurs at the point (1, 2, –1).

Holy Faith New Style Sample Paper–3

(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions : Same as Holy Faith New Style Sample Paper–1.

SECTION—A

(Multiple choice questions, each question carries 1 mark)

Q. 1. Let A = {3, 5}. Then number of reflexive relations on A is :

(*a*) 2 (*b*) 4

(*c*) 0 (*d*) 8.

Ans. (*b*) 4

Q. 2. The principal value of $\tan^{-1}\left(\tan\frac{9\pi}{8}\right)$ is :

(*a*) $\frac{\pi}{8}$ (*b*) $\frac{3\pi}{8}$

(*c*) $-\frac{\pi}{8}$ (*d*) $-\frac{3\pi}{8}$.

Ans. (*a*) $\frac{\pi}{8}$

Q. 3. If A and B are symmetric matrices of same order, then ($AB^t - 2BA^t$) is a :

(*a*) Skew-symmetric matrix

(*b*) Symmetric matrix

(*c*) Neither symmetric nor skew-symmetric matrix

(*d*) Null matrix.

Ans. (*c*) Neither symmetric nor skew-symmetric matrix

Q. 4. If A = $\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$ and 2A + B is a null matrix, then B is equal to :

(*a*) $\begin{bmatrix} 6 & 8 \\ 10 & 4 \end{bmatrix}$ (*b*) $\begin{bmatrix} -6 & -8 \\ -10 & -4 \end{bmatrix}$

(*c*) $\begin{bmatrix} 5 & 8 \\ 10 & 3 \end{bmatrix}$ (*d*) $\begin{bmatrix} -5 & -8 \\ -10 & -3 \end{bmatrix}$.

Ans. (*b*) $\begin{bmatrix} -6 & -8 \\ -10 & -4 \end{bmatrix}$

5. If A = $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and B^{-1} = $\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$, then $(AB)^{-1}$ is equal to :

(*a*) $\begin{bmatrix} 15 & -19 \\ -26 & 33 \end{bmatrix}$ (*b*) $\begin{bmatrix} 11 & -14 \\ -29 & 37 \end{bmatrix}$

(*c*) $\begin{bmatrix} 37 & 14 \\ 29 & 11 \end{bmatrix}$ (*d*) $\begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$

Ans. (*d*) $\begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$

Q. 6. The value of '*k*' (< 0) for which the function *f* defined as :

$$f(x) = \begin{cases} \dfrac{1-\cos kx}{x\sin x}, & x \neq 0 \\ \dfrac{1}{2}, & x = 0 \end{cases}$$

is continuous at $x = 0$ is :

(*a*) ± 1 (*b*) – 1

(*c*) $\pm\frac{1}{2}$ (*d*) $\frac{1}{2}$.

Ans. (*b*) – 1

Q. 7. The derivative of $\sin^{-1}(2x\sqrt{1-x^2})$ w.r.t. $\sin^{-1}x$, $\frac{1}{\sqrt{2}} < x < 1$, is :

(*a*) 2 (*b*) $\frac{\pi}{2} - 2$

(*c*) $\frac{\pi}{2}$ (*d*) – 2.

Ans. (*a*) 2

Q. 8. If $f(x) = a(x - \cos x)$ is strictly decreasing in R, then '*a*' belongs to :

(*a*) {0} (*b*) (0, ∞)

(*c*) (–∞, 0) (*d*) (–∞, ∞).

Ans. (*d*) (–∞, ∞).

Q. 9. $\int \frac{\cos 2x}{(\sin x + \cos x)^2}\,dx$ is equal to :

(*a*) $\frac{-1}{\sin x + \cos x}$

(*b*) log | sin *x* + cos *x* | + *c*.

(*c*) log | sin *x* – cos *x* | + *c*

(*d*) $\dfrac{1}{(\sin x + \cos x)^2}$.

Ans. (*b*) log | sin *x* + cos *x* | + *c*.

Q. 10.The value of $\int_{-\pi/2}^{\pi/2} x^3 \sin^4 x\, dx$ **is :**

(*a*) 0 (*b*) $\dfrac{\pi}{2}$

(*c*) π (*d*) $\dfrac{\pi}{2}$.

Ans. (*a*) 0

Q. 11. The area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **is equal to :**

(*a*) $\pi^2 ab$ (*b*) πab

(*c*) $\pi a^2 b$ (*d*) πab^2.

Ans. (*b*) πab

Q. 12. The sum of the order and the degree of the differential equation:

$$\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^3\right) = 0 \text{ is :}$$

(*a*) 2 (*b*) 3

(*c*) 5 (*d*) 0.

Ans. (*b*) 3

Q. 13. Kapila is trying to find the general solution of the following differential equations :

(*i*) $x\, e^{x/y} dx - y\, e^{3x/y}\, dy = 0$

(*ii*) $(2x+1)\dfrac{dy}{dx} = 3 - 2y$

(*iii*) $\dfrac{dy}{dx} = \sin x - \cos y$.

Which of the above becomes Variables Separable by substituting *y* = *bx*, where *b* is a variable ?

(*a*) Only (*i*) (*b*) Only (*i*) and (*ii*)

(*c*) All (*i*), (*ii*) and (*iii*) (*d*) None of these.

Ans. (*a*) Only (*i*)

Q. 14. Area of a rectangle having vertices :

A $\left(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$,B $\left(\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$,

C $\left(\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}\right)$,D $\left(-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}\right)$ is :

(*a*) $\dfrac{1}{2}$ square unit (*b*) 1 square unit

(*c*) 2 square units (*d*) 4 square units.

Ans. (*c*) 2 square units

Q. 15. The value of λ for which two vectors : $2\hat{i} - \hat{j} + 2\hat{k}$ **and** $3\hat{i} + \lambda\hat{j} + \hat{k}$ **are perpendicular is :**

(*a*) 2 (*b*) 4

(*c*) 6 (*d*) 8.

Ans. (*d*) 8

Q. 16. L_1 **and** L_2 **are two skew lines. How many lines joining** L_1 **and** L_2 **can be drawn such that the line is perpendicular to both** L_1 **and** L_2**.**

(*a*) exactly one

(*b*) exactly two

(*c*) infinitely many

(*d*) there cannot be a line joining two skew lines such that it is perpendicular to both.

Ans. (*a*) exactly one

Q. 17. The number of corner points of the feasible region determined by the constraints :

$x - y \geq 0,\ 2y \leq x + 2,\ x \geq 0,\ y \geq 0$ **is :**

(*a*) 2 (*b*) 3

(*c*) 4 (*d*) 5.

Ans. (*b*) 3

Q. 18. If A and B are two events such that $A \subset B$ **and P (B)** $\neq 0$**, then which of the following is correct :**

(*a*) $P(A/B) = \dfrac{P(B)}{P(A)}$ (*b*) P (A/B) < P (A)

(*c*) P (A/B) ≥ P (A) (*d*) None of these.

Ans. (*c*) P (A/B) ≥ P (A)

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.

(*b*) Both 'A' and 'R' are true but 'R' is not the correct explanation of 'A'.

(*c*) 'A' is true but 'R' is false.

(*d*) 'A' is false but 'R' is true.

Q. 19. Assertion (A) : $\displaystyle\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}\, dx = 3.$

Reason (R) : $\displaystyle\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx.$

Ans. (*c*) $\displaystyle I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}\, dx$ ....(1)

$\therefore \quad \displaystyle I = \int_2^8 \frac{\sqrt{10-(10-x)}}{\sqrt{10-x} + \sqrt{10-(10-x)}}\, dx$

$$= \int_0^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}\,dx \qquad ....(2)$$

Adding (1) and (2), $2I = \int_2^8 1\,dx$

$$= [x]_2^8 = 8 - 2 = 6.$$

$\Rightarrow$ I $= 3$.

Thus, (A) is true and (R) is also true and explains (A).

Q. 20. Assertion (A) : Area of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ is 16π sq. units.

Reason (R) : Area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab sq. units.

Ans. (*d*) Area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab, which is true.

But area of the ellipse is $\frac{x^2}{9}+\frac{y^2}{16}=1$ is $\pi(3)(4) = 12\pi$ sq. units, which is false.

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Find the value of :

$$\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right).$$

Solution. $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

$$= \cos^{-1}\left(\cos\frac{\pi}{3}\right)+2\sin^{-1}\left(\sin\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3}+2\left(\frac{\pi}{6}\right) \quad \left[\because \frac{\pi}{3}\in[0,\pi] \text{ and } \frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$$

$$= \frac{\pi}{3}+\frac{\pi}{3} = \frac{2\pi}{3}.$$

Or

If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then find the value of :

$$\alpha(\beta+\gamma) - \beta(\gamma+\alpha) + \gamma(\alpha+\beta).$$

Solution. We have : $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

$\Rightarrow \quad \cos^{-1}\alpha = \pi, \cos^{-1}\beta = \pi$ and $\cos^{-1}\gamma = \pi$

$\Rightarrow \quad \alpha = \cos\pi, \beta = \cos\pi$ and $\gamma = \cos\pi$

$\Rightarrow \quad \alpha = -1, \beta = -1$ and $\gamma = -1$.

Thus, $\alpha = \beta = \gamma = -1$.

$\therefore \alpha(\beta+\gamma) - \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$

$$= (-1)(-1-1) - (-1)(-1-1) + (-1)(-1-1)$$

$$= 2 - 2 + 2 = 2.$$

Q. 22. Show that the function f(x) = $\frac{16\sin x}{4+\cos x} - x$ is strictly decreasing in $\left(\frac{\pi}{2},\pi\right)$.

Solution. Here, $f(x) = \frac{16\sin x}{4+\cos x} - x$.

$$\therefore f'(x) = 16\left[\frac{(4+\cos x)\cos x - \sin x(0-\sin x)}{(4+\cos x)^2}\right]-1$$

$$= 16\left[\frac{4\cos x+\cos^2 x+\sin^2 x}{(4+\cos x)^2}\right]-1$$

$$= 16\left[\frac{4\cos x+1}{(4+\cos x)^2}\right]-1$$

$$= \frac{64\cos x+16-(4+\cos x)^2}{(4+\cos x)^2}$$

$$= \frac{56\cos x-\cos^2 x}{(4+\cos x)^2}.$$

Now $\frac{56\cos x-\cos^2 x}{(4+\cos x)^2} < 0$

$\Rightarrow \cos x\,(56 - \cos x) < 0$

$\Rightarrow \cos x < 0$, which is true in $\left(\frac{\pi}{2},\pi\right)$.

Q. 23. Find all the points of local maxima and local minima of the function 'f' given by :

$f(x) = 2x^3 - 6x^2 + 6x + 5$.

Solution. We have : $f(x) = 2x^3 - 6x^2 + 6x + 5$.

$\therefore \quad f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2$.

$\therefore \quad f''(x) = 12(x-1)$.

Now $f'(x) = 0$ gives $x = 1$.

Also, $f''(1) = 0$.

Thus $x = 1$ is neither a point of maxima nor of minima.

Now $f'''(x) = 12.$

And $f'''(x)\Big]_{x=1} = 12 \neq 0.$

Hence, $x = 1$ is a point of inflexion.

Or

What is the maximum value of the function : $\sin x + \cos x$?

Solution. We have :

$f(x) = \sin x + \cos x \; ; \; x \in [0, 2\pi]$ (say)

$\therefore \quad f'(x) = \cos x - \sin x.$

For extreme values, $f'(x) = 0$

$\Rightarrow \quad \cos x - \sin x = 0$

$\Rightarrow \quad \sin x = \cos x \quad \Rightarrow \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}.$

Now $f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

and $f(2\pi) = \sin 2\pi + \cos 2\pi = 0 + 1 = 1.$

Hence, max. value of $f(x) = \sqrt{2}$.

Q. 24. Evaluate : $\int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx.$

Solution. Let $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$ ...(1)

$$\therefore \quad I = \int_0^{2\pi} \frac{1}{1+e^{\sin(2\pi - x)}} dx \quad [\textit{Property V}]$$

$$= \int_0^{2\pi} \frac{1}{1+e^{-\sin x}} dx \quad \text{...(2)}$$

Adding (1) and (2), $2I = \int_0^{2\pi} \left(\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}}\right) dx$

$$= \int_0^{2\pi} \left(\frac{1}{1+e^{\sin x}} + \frac{e^{\sin x}}{1+e^{\sin x}}\right) dx$$

$$= \int_0^{2\pi} \left(\frac{1+e^{\sin x}}{1+e^{\sin x}}\right) dx$$

$$= \int_0^{2\pi} 1.dx = [x]_0^{2\pi}$$

$$= 2\pi - 0 = 2\pi.$$

Hence, $I = \pi,$

Q. 25. Find the value of 'k' so that the function f is continuous at x = 5 :

$$\mathbf{f(x)} = \begin{cases} \mathbf{kx + 1 \text{ if }} & \mathbf{x \le 5} \\ \mathbf{3x - 5 \text{ if }} & \mathbf{x > 5.} \end{cases}$$

Solution. $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (kx + 1)$

$$= \lim_{h \to 0} (k(5-h)+1) = k(5-0)+1 = 5k+1$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (3x - 5)$$

$$= \lim_{h \to 0} (3(5+h)-5) = 3(5+0)-5 = 10.$$

And $f(5) = k(5) + 1 = 5k + 1.$

For continuity at $x = 5,$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$$

$\Rightarrow \quad 5k + 1 = 10 = 5k + 1.$

Hence, $5k + 1 = 10$ *i.e.* $k = \frac{9}{5}.$

SECTION—C

(This section comparies of short answer type questions (SA) of 3 marks each)

Q. 26. Evaluate : $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$.

Solution. Let $I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$.

Put $\sin^{-1} x = t$ *i.e.* $x = \sin t$

so that $dx = \cos t \, dt.$

$$\therefore \; I = \int \frac{\sin t \, . \, t}{\sqrt{1-\sin^2 t}} \cos t \, dt$$

$$= \int \frac{t \sin t}{\cos t} . \cos t \, dt = \int t \sin t \, dt$$

$$= t(-\cos t) - \int (1)(-\cos t) \, dt$$

[*Integrating by parts*]

$$= -t \cos t + \int \cos t \, dt = -t \cos t + \sin t + c$$

$$= -t\sqrt{1-\sin^2 t} + \sin t + c$$

$$= -\sin^{-1} x . \sqrt{1-x^2} + x + c$$

$$= x - \sqrt{1-x^2}\ \sin^{-1} x + c.$$

Q. 27. A problem in Mathematics is given to three students whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. What is the probability in the following cases ?

(i) that the problem is solved

(ii) only one (exactly one) of them solves it correctly

(iii) atleast one of them may solve it.

Solution. Let A, B, C be three events when a problem in Mathematics is solved by three students.

Given : $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}, P(C) = \frac{1}{4}$.

$\therefore \quad P(\overline{A}) = 1 - \frac{1}{2} = \frac{1}{2}, P(\overline{B}) = 1 - \frac{1}{3} = \frac{2}{3}$

and $\quad P(\overline{C}) = 1 - \frac{1}{4} = \frac{3}{4}$.

(*i*) Probability that the problem is solved

= Probability that the problem is solved by atleast one student

$$= 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C})$$

$$= 1 - \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = 1 - \frac{1}{4} = \frac{3}{4}.$$

(*ii*) Probability that only one solves it correctly

$$= P(A \cap \overline{B} \cap \overline{C}) + P(\overline{A} \cap B \cap \overline{C}) + P(\overline{A} \cap \overline{B} \cap C)$$

$$= P(A)\, P(\overline{B})\, P(\overline{C}) + P(\overline{A})\, P(B)\, P(\overline{C}) + P(\overline{A})\, P(\overline{B})\, P(C)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} + \frac{1}{2} \times \frac{1}{3} \times \frac{3}{4} + \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4}$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{12} = \frac{11}{24}.$$

(*iii*) Probability that atleast one of them may solve the problem

$$= 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C})$$

$$= 1 - \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Q. 28. Find : $\int \frac{\mathbf{x^2}}{\mathbf{(x^2+1)(3x^2+4)}} \mathbf{dx}$ **.**

Solution. Put $x^2 = y$

$$\therefore \quad \frac{x^2}{(x^2+1)(3x^2+1)}$$

$$= \frac{y}{(y+1)(3y+4)} \equiv \frac{A}{y+1} + \frac{B}{3y+4} \qquad ....(1)$$

$\Rightarrow \quad y \equiv A(3y-4) + B(y+1)$.

Putting $y = -1$, $-1 = A(-3+4) \Rightarrow A = -1$.

Putting $y = -\frac{4}{3}$, $-\frac{4}{3} = B\left(-\frac{4}{3}+1\right)$

$\Rightarrow -\frac{4}{3} = B\left(-\frac{1}{3}\right) \Rightarrow B = 4$.

$\therefore$ From (1), $\frac{y}{(y+1)(3y+4)}$

$$= \frac{-1}{y+1} + \frac{4}{3y+4}$$

$$\Rightarrow \quad \frac{x^2}{(x^2+1)(3x^2+4)} = \frac{-1}{x^2+1} + \frac{4}{3x^2+4}.$$

$$\therefore \quad \int \frac{x^2}{(x^2+1)(3x^2+4)} dx$$

$$= -\int \frac{1}{1+x^2} dx + 4\int \frac{1}{3x^2+4} dx$$

$$= -\int \frac{1}{1+x^2} dx + \frac{4}{3}\int \frac{dx}{\left(\frac{2}{\sqrt{3}}\right)^2 + x^2}$$

$$= -\tan^{-1}\frac{x}{1} + \frac{4}{3}.\frac{1}{2/\sqrt{3}} \tan^{-1}\frac{x}{2/\sqrt{3}} + c$$

$$= -\tan^{-1}(x) + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + c.$$

Or

Evaluate : $\int_0^{\pi} \mathbf{e^{2x}. \sin\left(\frac{\pi}{4}+x\right) dx}$**.**

Solution. Let $\quad I = \int_0^{\pi} e^{2x}. \sin\left(\frac{\pi}{4}+x\right) dx.$

Put $\frac{\pi}{4} + x = t$

$\Rightarrow \quad x = t - \frac{\pi}{4}$ so that $dx = dt$.

When $\quad x = 0,\ t = \frac{\pi}{4}$. When $x = \pi$, $t = \frac{5\pi}{4}$.

$$\therefore \quad I = \int_{\pi/4}^{5\pi/4} e^{2t-\frac{\pi}{2}} \sin t\, dt$$

$$= e^{\frac{-\pi}{2}} \int_{\pi/4}^{\frac{5\pi}{4}} e^{2t} \sin t\, dt$$

$$= e^{\frac{-\pi}{2}} \int_{\pi/4}^{5\pi/4} \sin t\, e^{2t} dt$$

$$= e^{\frac{-\pi}{2}} \left[\left(\sin t \,.\, \frac{e^{2t}}{2}\right) - \int \cos t \,.\, \frac{e^{2t}}{2} dt\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

[*Integrating by parts*]

$$= e^{\frac{-\pi}{2}}$$

$$\left[\frac{1}{2}\left(\sin \frac{5\pi}{4} e^{\frac{5\pi}{2}} - \sin\frac{\pi}{4} e^{\frac{\pi}{2}}\right) - \left\{\cos t \,.\, \frac{e^{2t}}{4} + \int \sin t \,.\, \frac{e^{2t}}{4} dt\right\}_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\right]$$

[*Again integrating by parts*]

$$= e^{\frac{-\pi}{2}}$$

$$\left[\frac{1}{2}(e^{\frac{5\pi}{2}} \sin\frac{5\pi}{4} - e^{\frac{\pi}{2}} \sin\frac{\pi}{4}) - \left(\frac{e^{2t}}{4} \cos t\right)_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{e^{2t}}{4} \sin t\, dt\right]$$

$$= e^{\frac{-\pi}{2}}$$

$$\left[\frac{1}{2}\left(e^{\frac{5\pi}{2}} \sin \frac{5\pi}{4} - e^{\frac{\pi}{2}} \sin\frac{\pi}{4}\right) - \left\{\cos t \,.\, \frac{e^{2t}}{4} + \int \sin t \,.\, \frac{e^{2t}}{4} dt\right\}_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\right]$$

$$= e^{\frac{-\pi}{2}}$$

$$\left[\frac{1}{2}(e^{\frac{5\pi}{2}} \sin\frac{5\pi}{4} - e^{\frac{\pi}{2}} \sin\frac{\pi}{4}) - \left(\frac{e^{2t}}{4} \cos t\right)_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{e^{2t}}{4} \sin t\, dt\right]$$

$$= e^{\frac{-\pi}{2}}$$

$$\left[\frac{1}{2}\left(-\frac{1}{\sqrt{2}} e^{\frac{5\pi}{2}} - \frac{1}{\sqrt{2}} e^{\frac{\pi}{2}}\right) - \frac{1}{4}\left(-\frac{1}{\sqrt{2}} e^{5\pi/2} - \frac{1}{\sqrt{2}} e^{\pi/2}\right)\right] - \frac{I}{4}$$

$$\Rightarrow I + \frac{I}{4} = -\frac{1}{2\sqrt{2}} [e^{2\pi} + 1]$$

$$+ \frac{1}{4\sqrt{2}} [e^{2\pi} + 1]$$

$$\Rightarrow \frac{5}{4} I = \frac{(e^{2\pi} + 1)}{2\sqrt{2}} \left[\frac{1}{2} - 1\right]$$

$$= -\frac{1}{4\sqrt{2}} (e^{2\pi} + 1).$$

Hence, $I = -\dfrac{1}{5\sqrt{2}} (1 + e^{2\pi})$.

Q. 29. Solve : x dy – y dx = $\sqrt{x^2 - y^2}$ dx .

Solution. The given equation is :

$$x\, dy - y\, dx = \sqrt{x^2 - y^2}\, dx$$

$$\Rightarrow \quad x\, dy = \left(y + \sqrt{x^2 - y^2}\right) dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x} \qquad ...(1)$$

This is homogeneous in x and y.

Put y = vx ...(2)

so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ ...(3)

From (1), using (2) and (3), we get :

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 - v^2x^2}}{x}$$

$$\Rightarrow \quad v + x\frac{dy}{dx} = v + \sqrt{1 - v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1 - v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1 - v^2}} = \frac{dx}{x}.$$ |*Variables Separable*

Integrating, $\displaystyle\int \frac{1}{\sqrt{1 - v^2}} dv = \int \frac{dx}{x} + c$

$$\Rightarrow \quad \sin^{-1} v = \log |x| + c$$

$$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \log |x| + c,$$

which is the reqd. solution.

Or

Solve the differential equation :

$(1 + x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$, subject to the initial condition y(0) = 0.

Solution. The given equation can be written as:

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2} \quad ...(1) \text{ | } \textbf{\textit{Linear Equation}}$$

Comparing with $\frac{dy}{dx} + Py = Q$, we have:

$$\text{'P'} = \frac{2x}{1+x^2} \text{ and 'Q'} = \frac{4x^2}{1+x^2}.$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log|1+x^2|}$$

$$= e^{\log(1+x^2)} = (1+x^2).$$

Multiplying (1) by $(1 + x^2)$, we get:

$$(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$$

Intergrating, $y \cdot (1+x^2) = \int 4x^2 dx = \frac{4x^3}{3} + c$

$$\Rightarrow \qquad y(1+x^2) = \frac{4}{3}x^3 + c \qquad ...(2)$$

When $x = 0, y = 0$;

$\therefore \qquad c = 0 + c$

$\Rightarrow \qquad c = 0$.

Putting in (2), $y(1+x^2) = \frac{4}{3}x^3$,

which is the required solution.

Q. 30. Minimise and Maximise Z = x + 2y subject to $x + 2y \geq 100, 2x - y \leq 0, 2x + y \leq 200, x, y \geq 0$.

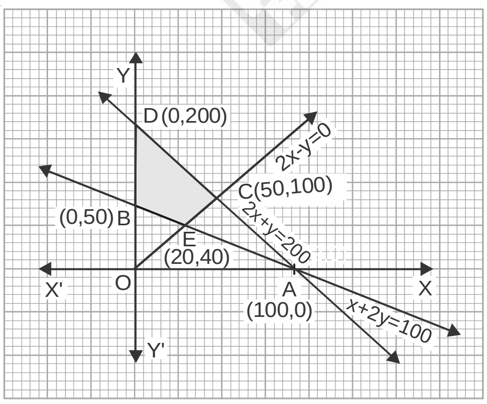
Solution : The system of constraints is :

$x + 2y \geq 100$ ...(1)

$2x - y \leq 0$ ...(2)

$2x + y \leq 200$ ...(3)

and $x, y \geq 0$ ...(4)



The shaded region in the above figure is the feasible region determined by the system of constraints (1)–(4). It is observed that the feasible region ECDB is bounded. Thus we use ***Corner Point Method*** to determine the maximum of Z, where :

$$Z = x + 2y \qquad ...(5)$$

The co-ordinates of E, C, D and B are

(20, 40)(on solving $x + 2y = 100$ and $2x - y = 0$)

(50, 100)(on solving $2x + y = 200$ and $2x - y = 0$)

(0, 200) and (0, 50) respectively.

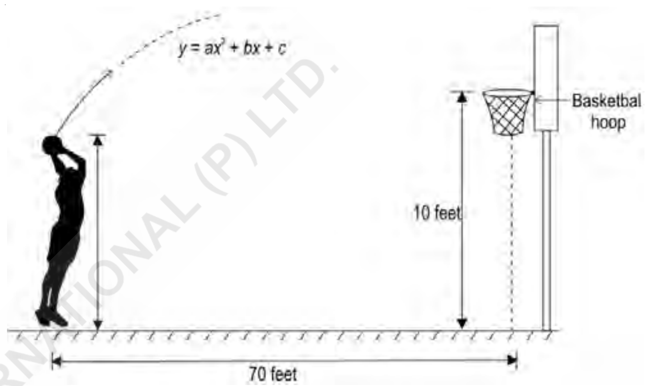
Corner point	**Corresponding Value of Z**
E : (20, 40)	**100 (Minimum)**
C : (50, 100)	250
D : (0, 200)	**400 (Maximum)**
B : (0, 50)	**100 (Minimum)**

Hence, $Z_{max} = 400$ at (0, 200)

and $Z_{min} = 100$ at all points on the line segment joining the points B (0, 50) and E (20, 40).

Or

Abdul threw a basketball in the direction of the basketball hoop which traversed a parabolic path in a vertical plane as shown below :



The equation of the path traversed by the ball is y = ax^2 + bx + c with respect to a xy-coordinate system in the vertical plane. The ball traversed through the points (10, 16), (20, 22) and (30, 25). The basketball hoop is at a horizontal distance of 70 feet from Abdul. The height of the basketball hoop is 10 feet from the floor to the top edge of the rim.

Did the ball successfuly go through the hoop ? Justify your answer.

Solution. The equation of the path tranversed by the ball is :

$$y = ax^2 + bx + c \qquad ....(1)$$

Since the points (10, 16) : (20, 22) and (30, 25) lie on (1),

$\therefore \quad 16 = a(100) + b(10) + c \Rightarrow 100a + 10b + c = 16$ ...(2)

$22 = a(400) + b(20) + c \Rightarrow 400a + 20b + c = 22$ ...(3)

and $25 = a(900) + b(30) + c \Rightarrow 900a + 30b + c = 25$ ...(4)

Writing the above system of equation in the form AX = B,

where $A = \begin{bmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 900 & 30 & 1 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $B = \begin{bmatrix} 16 \\ 22 \\ 25 \end{bmatrix}$.

Now $|A| = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 900 & 30 & 1 \end{vmatrix}$

$= 1.(12000 - 18000) - 1.(3000 - 9000) + 1.(2000 - 4000)$

$= -6000 + 6000 - 2000 = -2000 \neq 0$

$\Rightarrow A^{-1}$ exists.

As usual, $A^{-1} = \begin{pmatrix} \frac{1}{200} & -\frac{1}{100} & \frac{1}{200} \\ -\frac{1}{4} & \frac{2}{5} & -\frac{3}{20} \\ 3 & -3 & 1 \end{pmatrix}$

Solving, we get $a = -\frac{3}{200}, b = \frac{21}{20}$ and $c = 7$.

From (1), $X = A^{-1}B$

$$\Rightarrow \begin{pmatrix} -\frac{3}{200} \\ \frac{21}{20} \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{200} & -\frac{1}{100} & \frac{1}{200} \\ -\frac{1}{4} & \frac{2}{5} & -\frac{3}{20} \\ 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} 16 \\ 22 \\ 25 \end{pmatrix}.$$

Thus the equation of the path transvered by the ball is :

$$y = -\frac{3}{200}x^2 + \frac{21}{20}x + 7.$$

When $x = 70$ feet,

$$\text{Then} \quad y = -\frac{3}{200}(4900) + \frac{21}{20}(70) + 7$$

$$= -\frac{147}{2} + \frac{147}{2} + 7 = 7 \text{ feet.}$$

Thus the ball went by 7 feet above the floor, which means 3 feet below the basketball hoop.

Hence, the ball did not go through the hoop.

Q. 31. Determine graphically the minimum value of the objective function :

Z = – 50x + 20y

subject to the constraints :

2x – y ≥ – 5, 3x + y ≥ 3, 2x – 3y ≤ 12, x , y ≥ 0.

Solution. The system of constraints is :

$2x - y \geq -5$ ...(1)

$3x + y \geq 3$ ...(2)

$2x - 3y \leq 12$ ...(3)

and $x, y \geq 0$ ...(4)

The shaded region in the following figure is the feasible region determined by the system of constraints (1) – (4).

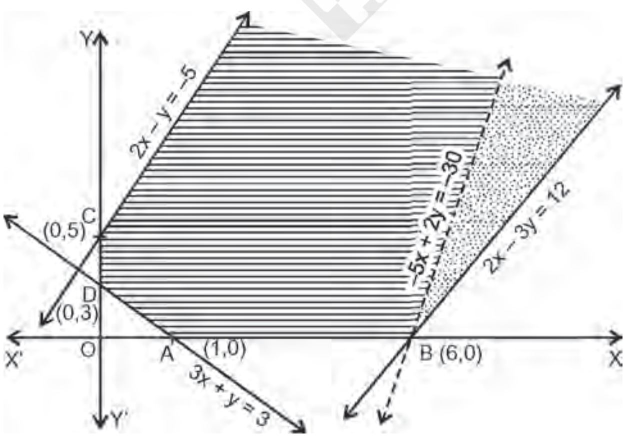


Fig.

It is observed that the feasible region is unbounded.

We evaluate $Z = -50x + 20y$ at the corner points :

A (1, 0), B (6, 0), C (0, 5) and D (0, 3) :

Corner Point	Corresponding Value of Z
A : (1, 0)	– 50
B : (6, 0)	**– 300 (Minimum)**
C : (0, 5)	100
D : (0, 3)	60

From the table, we observe that – 300 is the minimum value of Z.

But the feasible region is unbounded.

∴ – 300 may or may not be the minimum value of Z.

For this, we draw the graph of the inequality.

$-50x + 20y < -300$

i.e. $-5x + 2y < -30$.

Since the remaining half-plane has common points with the feasible region,

∴ $Z = -50x + 20y$ has no minimum value.

SECTION—D

(This section comprises of Long Answer Type questions (LAQ) of 5 marks each)

Q. 32. Find the area of the region bounded by the curve $y^2 = 4x$, y - axis and line y = 3.

Sol. $y^2 = 4x$ is right-handed parahola.

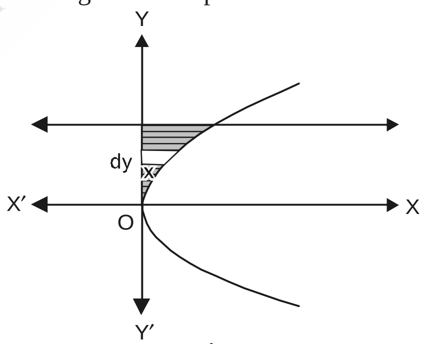


Fig.

$$\text{Reqd. area} = \int_0^3 x\,dy \quad \text{[\textit{Taking horizonntal strips}]}$$

$$= \int_0^3 \frac{y^2}{4}dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3$$

$$= \frac{1}{12}[27 - 0] = \frac{27}{12} = \frac{9}{4} \text{ sq. units.}$$

Q. 33. Show that the relation S on the set :

A = {x ∈ Z : 0 ≤ x ≤ 12} given by:

S = {(a, b) : a, b ∈ Z, |a – b| is divisible by 3}

is an equivalence relation.

Solution. Reflexive : $(a, a) \in$ **S**

$\Rightarrow |a - a|$ *i.e.*, | 0 | is divisible by 3.

Thus, S is reflexive.

Symmetric :

$(a, b) \in S \Rightarrow |a - b|$ is divisible by 3

$\Rightarrow |b - a|$ is divisible by 3

$\Rightarrow (b, a) \in S$.

Thus, S is symmetric.

Transitive :

Let $(a, b) \in$ S and $(b, c) \in$ S.

Thus $|a - b| = 3m$ and $|b - c| = 3n$

$\Rightarrow$ $a - b = \pm 3m$ and $b - c = \pm 3n$.

$\therefore$ $(a - c) = 3(\pm m \pm n)$

$\Rightarrow a - c$ is divisible by 3

$\Rightarrow |a - c|$ is divisible by 3 $\Rightarrow (a, c) \in$ S.

Thus, S is transitive.

Hence, S is an equivalence relation.

Or

A function f : R – {– 1, 1} → R is defined by f (x) = $\mathbf{\frac{x}{x^2 - 1}}$.

(i) Check if f is one-one. (ii) Check if f is onto. Show your work.

Solution. **(*i*)** Assume that $f(x) = f(y)$.

$$\therefore \quad \frac{x}{x^2 - 1} = \frac{y}{y^2 - 1}$$

$\Rightarrow x(y^2 - 1) = y(x^2 - 1)$

$\Rightarrow xy^2 - x = yx^2 - y$

$\Rightarrow xy^2 - x - yx^2 + y = 0$

$\Rightarrow xy(y - x) + (y - x) = 0 \Rightarrow (y - x)(xy + 1) = 0$

$\Rightarrow y - x = 0$ or $xy + 1 = 0$

$\Rightarrow$ either $x = y$ or $xy = -1$.

Let us take numbers x and y such that $xy = -1$.

Hence, f is not one-one.

For Example : Take $x = \frac{1}{2}$ and $y = -2$.

$$\therefore \quad f(x) = \frac{1/2}{\frac{1}{n} - 1} = \frac{1/2}{3/4} = -\frac{2}{3}$$

and $$f(y) = \frac{-2}{4 - 1} = -\frac{2}{3}.$$

(*ii*) Here $y = f(x)$

$$\Rightarrow \frac{x}{x^2 - 1} = y \Rightarrow x = yx^2 - y$$

$$\Rightarrow yx^2 - x - y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 + 4y^2}}{2y}.$$

Since $1 + 4y^2 > 0$,

$\therefore$ real roots exist and also, these are not ± 1.

$$\therefore \quad x = \frac{1 \pm \sqrt{1 + 4y^2}}{2y} \in \mathbf{R} - \{-1, 1\}.$$

$\therefore$ For any $y \in \mathbf{R}$ (co-domain).

there exists $x \in \mathbf{R} - \{-1, 1\}$ (Domain)

such that $f(x) = y$.

Hence, f is onto.

Q. 34. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3, |\vec{b}| = 5$ and $|\vec{c}| = 7$, then find the angle between $\vec{a}$ and $\vec{b}$.

Solution. We have : $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$

$\therefore (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = (-\vec{c}).(-\vec{c})$

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = \vec{c}.\vec{c}$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = |\vec{c}|^2$

$\Rightarrow 2|\vec{a}||\vec{b}|\cos\theta = |\vec{c}|^2 - |\vec{a}|^2 - |\vec{b}|^2$

$\Rightarrow 2(3)(5)\cos\theta = 49 - 9 - 25$

$\Rightarrow \cos\theta = \frac{15}{30} = \frac{1}{2}$.

Hence, $\theta = \frac{\pi}{3}$.

Or

Find the value of 'λ', so that the lines:

$$\mathbf{\frac{1 - x}{3} = \frac{7y - 14}{\lambda} = \frac{z - 3}{2}} \text{ and } \mathbf{\frac{7 - 7x}{3\lambda} = \frac{y - 5}{1} = \frac{6 - z}{5}}$$

are at right angles. Also, find whether the lines are intersecting or not.

Solution. The given lines are :

$$\frac{x - 1}{-3} = \frac{y - 2}{\lambda/7} = \frac{z - 3}{2} \quad ...(1)$$

and $$\frac{x - 1}{-\frac{3\lambda}{7}} = \frac{y - 5}{1} = \frac{z - 6}{-5} \quad ...(2)$$

These are perpendicular if :

$$(-3)\left(-\frac{3\lambda}{7}\right) + \frac{\lambda}{7}(1) + 2(-5) = 0$$

if $\frac{9\lambda}{7} + \frac{\lambda}{7} - 10 = 0$ if $\frac{10\lambda}{7} = 10$.

Hence, $\lambda = 7$.

The direction-cosines of line (1) are < –3, 1, 2 >

The direction-cosines of line (2) are < –3, 1, –5 >.

Hence, the lines are intersecting.

Q. 35. A and B throw a pair of dice alternately. A wins the game if he gets a total of 7 and B wins if he gets a total of 10. If A starts the game, find the probability that B wins.

Solution. Number of outcomes = $6 \times 6 = 36$.

Favourable outcomes for winning of A are :

{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}.

$\therefore$ P(A wins) = P(A) = $\frac{6}{36} = \frac{1}{6}$

and $\quad P(\text{A loses}) = P(\bar{A}) = 1 - \frac{1}{6} = \frac{5}{6}$.

Favourable outcomes for winning of B are :

{(4, 6), (6, 4), (5, 5)}

$\therefore \quad P(\text{B wins}) = P(B) = \frac{3}{36} = \frac{1}{12}$

and $\quad P(\text{B loses}) = P(\bar{B}) = 1 - \frac{1}{12} = \frac{11}{12}$.

$$P(\text{B wins}) = P(\bar{A})\,P(B) + P(\bar{A})\,P(\bar{B})\,P(\bar{A})\,P(B) + P(\bar{A})\,P(\bar{B})\,P(\bar{A})\,P(\bar{B})\,P(\bar{A})\,P(B) + \ldots$$

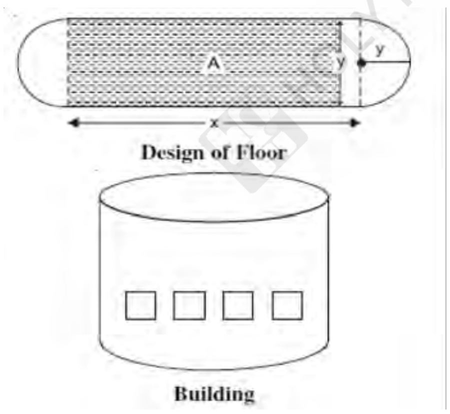
$$= \frac{5}{6}\times\frac{1}{12} + \frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{1}{12} + \frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{11}{12}\times\frac{5}{6}\times\frac{1}{12} \qquad |\,\textbf{G.P.}$$

$$= \frac{\frac{5}{72}}{1-\frac{5}{6}\times\frac{11}{12}} = \frac{5}{72-55} = \frac{5}{17}.$$

SECTION—E

(This section comprises of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36. An architect designs a building for a multi-national company. The floor consists of a rectangle region with semi-circular ends having a perimeter of 200 m is shown below :



Based on the above information, answer the following :

(*a*) If x and y represent the length and breadth of the rectangular region, then the relation between the variables :

(*b*) The CEO of the multiple-national company is interested in maximizing the area of the whole floor including the semi-circular ends. For this to happen find the value of x :

Solutions

(*a*) Perimeter$= 2x + 2\left(\pi \frac{y}{2}\right) = 2x + \pi y$

$$2x + \pi y = 200 \qquad \ldots(1)$$

(*b*) Here $A = xy + \pi\left(\frac{y}{2}\right)^2 \qquad \ldots(2)$

$$= xy + \frac{\pi}{4}y^2$$

$$= x\left(\frac{200-2x}{\pi}\right) + \frac{\pi}{4}\left(\frac{200-2x}{\pi}\right)^2 \qquad [\textit{Using (1)}]$$

$$= \frac{1}{\pi}(200-2x^2) + \frac{1}{\pi}(100-x)^2$$

$$= \frac{1}{\pi}(200x - 2x^2 + 10000 + x^2 - 200x)$$

$$= \frac{1}{\pi}(10000 - x^2)$$

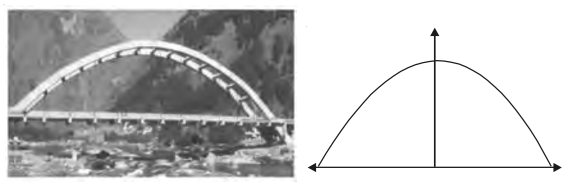
$$\therefore \quad \frac{dA}{dx} = \frac{1}{\pi}(-2x)$$

For max. A, $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{1}{\pi}(-2x) = 0$$

$$\Rightarrow \quad x = 0 \text{ m}$$

Q. 37.



The bridge connects two hills 100 feet apart. The arch on the bridge is in a parabolic form. The highest point on the bridge is 10 feet above the road at the middle of the bridge as seen in the figure.

Based on the information given above, answer the following questions :

(I) Find the equation of the parabola designed on the bridge.

(II) Find the value of the integral $\int_{-50}^{50} \frac{x^2}{250}\,dx$.

(III) Find the area formed by the curve $x^2 = 250y$, x-axis, $y = 0$ and $y = 10$.

Or

The area formed between $x^2 = 250y$, y-axis, $y = 2$ and $y = 4$.

Ans. (I) $x^2 = -250y$.

(II) $I = \dfrac{2}{250}\displaystyle\int_0^{50} x^2 dx = \dfrac{2}{250}\left[\dfrac{x^3}{3}\right]_0^{50}$

$= \dfrac{2}{750}[50 \times 50 \times 50] = \dfrac{1000}{3}$.

(III) Area $= \displaystyle\int_0^{10} x\,dy = \int_0^{10} \sqrt{250y}\;dy = \sqrt{250}\left[\dfrac{y^{3/2}}{3/2}\right]_0^{10}$

$= \dfrac{2}{3}\sqrt{250}\left[10^{3/2}\right] = \dfrac{2}{3} \times 5\sqrt{10}\left[10^{3/2}\right]$

$= \dfrac{2}{3} \times 5 \times 100 = \dfrac{1000}{3}$.

Or

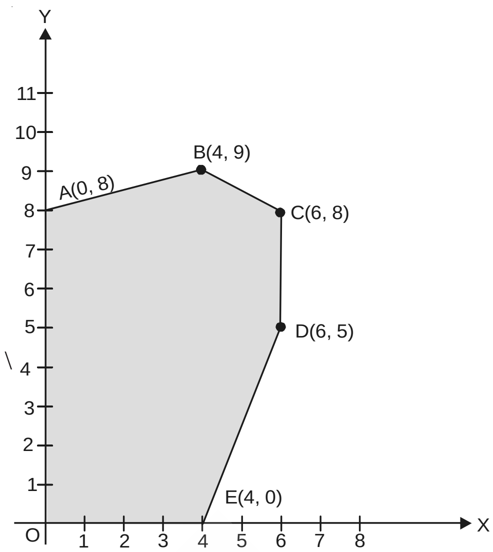
Area $= \displaystyle\int_2^4 x\,dy$

$= \displaystyle\int_2^4 \sqrt{250}\sqrt{y}\;dy = 5\sqrt{10}\left[\dfrac{y^{3/2}}{3/2}\right]_2^4$

$= \dfrac{10\sqrt{10}}{3}[4^{3/2} - 2^{3/2}]$

$= \dfrac{10\sqrt{10}}{3}[8 - 2\sqrt{2}]$.

Q. 38. The corner points of the feasible region determined by the system of linear constraints are as shown below :



Objective function is Z = $3x - 4y$.

Based on the above information, answer the following :

(I) Find maximum value of Z.

(II) Find minimum value of Z.

(III) Maximum value of Z occurs at:

(*a*) (4, 0) **(*b*) (6, 5)**

(*c*) (6, 8) **(*d*) (0, 8)**

OR

Maximum value occurs at which point ?

Ans. Objective function is Z = $3x - 4y$.

$Z_A = -32$, $Z_B = -24$, $Z_C = -14$,

$Z_D = -2$ and $Z_E = 12$.

(I) $Z_{max.} = 12$

(II) $Z_{min.} = -32$

(III) $Z_{max.}$ at E (4, 0)

OR

Max value occurs at (4, 0).

Holy Faith New Style Sample Paper–4

(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions : Same as Holy Faith New Style Sample Paper–1.

SECTION—A

(Multiple choice questions, each question carries 1 mark)

1. Let set X = {1, 2, 3} and a relation R is defined in X as : R = {(1, 3), (2, 2), (3, 2)}, then minimum ordered pairs which should be added in relation R to make it reflexive and symmetric are :

(*a*) {(1, 1), (2, 3), (1, 2)}
(*b*) {(3, 3), (3, 1), (1, 2)}
(*c*) {(1, 1), (3, 3), (3, 1), (2, 3)}
(*d*) {(1, 1), (3, 3), (3, 1), (1, 2)}.

Ans. (*c*) {(1, 1), (3, 3), (3, 1), (2, 3)}

2. The simplest form of $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$ is :

(*a*) $\frac{\pi}{4}-\frac{x}{2}$ (*b*) $\frac{\pi}{4}+\frac{x}{2}$

(*c*) $\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$ (*d*) $\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x$.

Ans. (*c*) $\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$

3. If A = $\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$, $B=\begin{bmatrix}x & 0\\1 & 1\end{bmatrix}$ and A = B^2, then '*x*' equals :

(*a*) ± 1 (*b*) – 1
(*c*) 1 (*d*) 2.

Ans. (*c*) 1

4. For any 2 × 2 matrix P, which of the following matrices can be Q such that PA = QP ?

(*a*) [1]

(*b*) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

(*c*) $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

(*d*) No such matrix exists as matrix multiplication is not commutative.

Ans. (*d*) No such matrix exists as matrix multiplication is not commutative.

5. If A and B are invertible square matrices of the same order, then which of the following is not correct ?

(*a*) adj.A = | A |·A^{-1} (*b*) det.(A)$^{-1}$ = [det.(A)$^{-1}$]
(*c*) (AB)$^{-1}$ = B^{-1}A^{-1} (*d*) (A + B)$^{-1}$ = B^{-1} + A^{-1}.

Ans. (*d*) (A + B)$^{-1}$ = B^{-1} + A^{-1}

6. If $f(x) = \log(3x + 1)$, then the value of $f''(1)$ is :

(*a*) $\frac{9}{16}$ (*b*) $-\frac{9}{16}$

(*c*) $\frac{9}{4}$ (*d*) $-\frac{9}{4}$.

Ans. (*b*) $-\frac{9}{16}$

7. If $f(x) = \begin{cases}\frac{x^2-9}{x-3}, & \text{if } x \neq 3\\ m, & \text{if } x = 3\end{cases}$ is continuous at x = 1, then the value of '*m*' is :

(*a*) 3 (*b*) 6
(*c*) 2 (*d*) 1.

Ans. (*b*) 6

8. The interval in which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing, is :

(*a*) (–1, ∞) (*b*) (–2, –1)
(*c*) (–∞, –2) (*d*) [–1, 1].

Ans. (*c*) (–∞, –2)

9. $\int \tan^{-1}\sqrt{x}\,dx$ is equal to :

(*a*) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+c$

(*b*) $x\tan^{-1}\sqrt{x}-\sqrt{x}+c$

(*c*) $\sqrt{x}-x\tan^{-1}\sqrt{x}+c$

(*d*) $\sqrt{x}-(x+1)\tan^{-1}\sqrt{x}+c$.

Ans. (*a*) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+c$

10. The value of $\int_0^{\pi/4} \sec 2x \, dx$ is:

(*a*) 0 (*b*) 1

(*c*) $\frac{1}{2}$ (*d*) $-\frac{1}{2}$.

Ans. (*b*) 1

11. Area lying between the curves $y^2 = 4x$ and $y = 2$ is :

(*a*) $\frac{2}{3}$ (*b*) $\frac{1}{3}$

(*c*) $\frac{1}{4}$ (*d*) $\frac{3}{4}$.

Ans. (*b*) $\frac{1}{3}$

12. The sum of the order and the degree of the differential equation:

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = \sin y \text{ is:}$$

(*a*) 5 (*b*) 2

(*c*) 3 (*d*) 4.

Ans. (*c*) 3

13. In which of the following differential equations is the degree equal to its order ?

(*a*) $x^3\left(\frac{dy}{dx}\right) - \frac{d^3y}{dx^3} = 0$

(*b*) $\left(\frac{d^3y}{dx^3}\right)^3 + \sin\left(\frac{dy}{dx}\right) = 0$

(*c*) $x^2\left(\frac{dy}{dx}\right)^4 + \sin y - \left(\frac{d^2y}{dx^2}\right)^2 = 0$

(*d*) $\left(\frac{dy}{dx}\right)^3 + x\left(\frac{d^2y}{dx^2}\right) - y^3 \frac{d^3y}{dx^3} + y = 0.$

Ans. (*c*) $x^2\left(\frac{dy}{dx}\right)^4 + \sin y - \left(\frac{d^2y}{dx^2}\right)^2 = 0$

14. The magnitude of the vector $6\hat{i} - 2\hat{j} + 3\hat{k}$ is :

(*a*) 1 (*b*) 5

(*c*) 7 (*d*) 12.

Ans. (*c*) 7

15. For what value of 'λ' the projection of vector $\hat{i} + \lambda\hat{j}$ on vector $\hat{i} - \hat{j}$ is $\sqrt{2}$?

(*a*) –1 (*b*) 1

(*c*) 0 (*d*) 3.

Ans. (*a*) –1

16. If the direction cosines of a line are : $< \frac{1}{c}, \frac{1}{c}, \frac{1}{c} >$, then :

(*a*) $0 < c < 1$ (*b*) $c > 2$

(*c*) $c = \pm\sqrt{2}$ (*d*) $c = \pm\sqrt{3}$.

Ans. (*d*) $c = \pm\sqrt{3}$

17. The corner points of the feasible region in the graphical representation of a linear programming problem are (2, 72), (15, 20) and (40, 15). If Z = 18*x* + 9*y* be the objective function, then :

(*a*) Z is maximum at (2, 72), minimum at (15, 20)
(*b*) Z is maximum at (15, 20), minimum at (40, 15)
(*c*) Z is maximum at (40, 15), minimum at (15, 20)
(*d*) Z is maximum at (40, 15), minimum at (2, 72).

Ans. (*c*) Z is maximum at (40, 15), minimum at (15, 20)

18. If A and B are two independent events such that P(A) = $\frac{1}{3}$ and P(B) = $\frac{1}{4}$, then P(B′/A) is:

(*a*) $\frac{1}{4}$ (*b*) $\frac{1}{8}$

(*c*) $\frac{3}{4}$ (*d*) 1.

Ans. (*c*) $\frac{3}{4}$

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.
(*b*) Both 'A' and 'R' are true but 'R' is not the correct explanation of 'A'.
(*c*) 'A' is true but 'R' is false.
(*d*) 'A' is false but 'R' is true.

Q. 19. Consider the system of equations :

$x - 2y + 3z = -1$
$-x + y - 2z = k$
$x - 3y + 4z = 1.$

Assertion (A) : Th system of equations has no solution for $k \neq 3$.

Reason (R) : The determinant

$$\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0 \text{ for } k \neq 3.$$

Ans. (*a*); Here, D = $\begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

$= (1)(4-6) + 2(-4+2) + 3(3-1)$

$= -2 - 4 + 6 = 0$

Similarly $D_1 = 3 - k$, $D_2 = k - 3$ and $D_3 = k - 3$

Clearly $D_1 = D_2 = D_3 = 0$ if $k = 3$.

Thus, the system of equations has no solution if $k \neq 3$.

And $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} = 3 - k = 0$ if $k = 3$.

Both Assertion (A) and Reason (R) are true and Reason (R) also explains assertion (A).

Q. 20. Assertion (A) : If P(A/B) ≥ P(A), then P(B/A) ≥ P(B).

Reason (R) : P(A/B) = $\dfrac{P(A \cap B)}{P(B)}$.

Ans. (A) $P(A \geq B) \geq P(A)$

$$\Rightarrow \quad \frac{P(A \cap B)}{P(B)} \geq P(A)$$

$$\Rightarrow \quad \frac{P(B \cap A)}{P(A)} \geq P(B)$$

$$\Rightarrow \quad P(B \cap A) \geq P(B).$$

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Find the value of

$$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}).$$

Solution. $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$$= \frac{\pi}{3} - \left(\pi - \frac{\pi}{6}\right) = -\frac{\pi}{2}.$$

Or

Evaluate : $\tan^{-1}\left(2\cos\left(2\sin^{-1}\left(\frac{1}{2}\right)\right)\right)$.

Solution. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

$$= \tan^{-1}\left[2\cos\left(2.\frac{\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[2\cos\frac{\pi}{3}\right] = \tan^{-1}\left(2.\frac{1}{2}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4}.$$

Q. 22. The population of rabbits in a forest is modelled by the function below :

P (t) = $\dfrac{2000}{1 + e^{-0.5t}}$**, where P represent the population of rabbits in t years. Determine whether the rabbit population is increasing or not, and justify your answer.**

Solution. We have :

$$P(t) = \frac{2000}{1 + e^{-0.5t}}.$$

$$\therefore \quad P'(t) = 2000 \times \frac{(-1)}{(1+e^{-0.5t})^2} \times e^{-0.5t} \times \left(-\frac{1}{2}\right)$$

$$= \frac{1000\,(e^{-0.5t})}{(1+e^{-0.5t})^2} > 0 \text{ for any value of } t.$$

Hence, the rabbit population is increasing.

Q. 23. Find the maximum profit that a company can make, if the profit function is given by :

P(x) = 72 + 42x – x²,

where x is the number of units and P is the profit in rupees.

Solution. Here, $P(x) = 72 + 42x - x^2$.

$\therefore \quad P'(x) = 42 - 2x$.

For maxima and minima,

$P'(x) = 0$

$\Rightarrow \quad 42 - 2x = 0$

$\Rightarrow \quad x = 21.$

And $\quad P''(x) = -2$ (–ve)

Thus, $P(x)$ is maximum when $x = 21$.

Hence, maximum value of $P = 72 + 42(21) - (21)^2$

$= 72 + 882 - 441$

$= 954 - 441 = 513.$

Hence, maxima profit is ₹ 513.

Or

Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

[Solution : Refer Q. 16 ; Ex. 6(*c*)]

Q. 24. Evaluate : $\displaystyle\int_{0.5}^{3.5} [x]\,dx$.

Solution. Here $f(x) = [x] = \begin{cases} 0, & 0\cdot5 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & 2 \leq x < 3 \\ 3, & 3 \leq x < 3\cdot5. \end{cases}$

$$\therefore \int_{0.5}^{3.5} [x]\,dx = \int_{0.5}^{1} [x]\,dx + \int_{1}^{2} [x]\,dx + \int_{2}^{3} [x]\,dx + \int_{3}^{3.5} [x]\,dx$$

[*Property IV Ext.*]

$$= \int_{0.5}^{1} 0.dx + \int_{1}^{2} 1.dx + \int_{2}^{3} 2.dx + \int_{3}^{3.5} 3dx$$

$$= 0 + \{x\}_1^2 + 2\{x\}_2^3 + 3\{x\}_3^{3.5}$$

$= (2 - 1) + 2\,(3 - 2) + 3\,(3{\cdot}5 - 3)$
$= 1 + 2 + 1{\cdot}5 = 4{\cdot}5.$

Q. 25. Examine the derivability of :

$$\mathbf{f\,(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad at\ x = 0.}$$

Solution. We have $D_f = \mathbf{R}$.

Now $$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h} = \lim_{h \to 0}\left(h \sin\frac{1}{h}\right) \qquad [\because h \neq 0]$$

$$= \lim_{h \to 0} h \,.\, \lim_{h \to 0} \sin\frac{1}{h} = 0.$$

$$\left[\because \lim_{h \to 0} h = 0 \text{ and } \sin\frac{1}{h} \text{ is bounded } \forall\ h \in \mathbf{R}, h \neq 0\right]$$

Hence, '*f*' is derivable at $x = 0$.

SECTION—C

(This section comprises of short answer type questions (SAQ) of 3 marks each)

Q. 26. Integrate : $\displaystyle \int \mathbf{e^x \left(\frac{x^2+1}{(x+1)^2}\right) dx}$.

Solution. $$\int e^x \left(\frac{x^2+1}{(x+1)^2}\right) dx$$

$$= \int \boldsymbol{e^x \left(\frac{x^2 - 1 + 2}{(x+1)^2}\right)} \qquad \textbf{(\textit{Note this step})}$$

$$= \int e^x \left(\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\frac{(x-1)(x+1)}{(x+1)^2} + \frac{2}{(x+1)^2}\right) dx$$

$$= \int e^x \left(\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right) dx$$

$$\left[\text{"}\mathbf{Form:} \int e^x\,[f(x) + f'(x)]\,dx\text{"}\right]$$

$$= \int \frac{x-1}{x+1}\,.\,e^x\,dx + \int \frac{2}{(x+1)^2}\,e^x\,dx$$

$$= \frac{x-1}{x+1}\,.\,e^x - \int \frac{(x+1)(1-0)-(x-1)(1+0)}{(x+1)^2}\,e^x\,dx$$

$$+ \int \frac{2}{(x+1)^2}\,e^x\,dx$$

[*Integrating first integral by parts*]

$$= \frac{x-1}{x+1}\,e^x - \int \frac{2}{(x+1)^2}\,e^x\,dx + \int \frac{2}{(x+1)^2}\,e^x\,dx$$

$$= \frac{x-1}{x+1}\,e^x + c.$$

Q. 27. An insurance company insured 2000 cyclists, 4000 scooter drivers and 6000 motorbike drivers. The probability of an accident involving a cyclist, scooter driver and a motorbike driver is 0·01, 0·03, 0·15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver.

Solution. Let the events be as below :

E_1 : Insured person is a cyclist

E_2 : Insured person is a scooter driver

E_3 : Insured person is a motorbike driver and

A : Insured person meets with an accident.

$$\therefore \quad P(E_1) = \frac{2000}{2000 + 4000 + 6000}$$

$$= \frac{2}{2+4+6} = \frac{2}{12} = \frac{1}{6},$$

$$P(E_2) = \frac{4000}{2000 + 4000 + 6000}$$

$$= \frac{4}{2+4+6} = \frac{4}{12} = \frac{1}{3}$$

and $$P(E_3) = \frac{6000}{2000 + 4000 + 6000}$$

$$= \frac{6}{2+4+6} = \frac{6}{12} = \frac{1}{2}.$$

Also $P(A/E_1) = 0{\cdot}01 = \dfrac{1}{100}$; $P(A/E_2) = 0{\cdot}03 = \dfrac{3}{100}$;

$$P(A/E_3) = 0{\cdot}15 = \frac{15}{100}.$$

By *Bayes' Theorem,*

$P(E_2/A)$

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\left(\frac{1}{3}\right)\left(\frac{3}{100}\right)}{\left(\frac{1}{6}\right)\left(\frac{1}{100}\right)+\left(\frac{1}{3}\right)\left(\frac{3}{100}\right)+\left(\frac{1}{2}\right)\left(\frac{15}{100}\right)}$$

$$= \frac{1}{\frac{1}{6}+1+\frac{15}{2}} = \frac{6}{1+6+45} = \frac{6}{52} = \frac{3}{26}.$$

Example 28. Find : $\int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx.$

Solution. Let $I = \int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

$$= \int \log(\log x)\,dx + \int \frac{1}{(\log x)^2}\,dx \quad ....(1)$$

Now $I_1 = \int \log(\log x)\,dx$

$= \int \log(\log x).1\,dx$ [*Guidance (3)*]

$= \log(\log x).x - \int \frac{1}{\log x . x}.x\,dx$

[*Integrating by parts*]

$= x\log(\log x) - \int \frac{1}{\log x}\,dx$ ...(2)

And, $I_2 = \int \frac{1}{\log x}\,dx$

$= \int \frac{1}{\log x}.1\,dx$ [*Guidance (3)*]

$= \frac{1}{\log x}.x - \int -\left(\frac{1}{\log x}\right)^2.\frac{1}{x}.x\,dx$

[*Integrating by parts*]

$= \frac{x}{\log x} + \int \frac{1}{(\log x)^2}\,dx.$

Putting in (2),

$$I_1 = x\log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2}\,dx.$$

Putting in (1),

$$I = x\log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2}\,dx + \int \frac{1}{(\log x)^2}\,dx$$

$$= x\log(\log x) - \frac{x}{\log x} + c$$

$$= x\left(\log(\log x) - \frac{1}{\log x}\right) + c.$$

Or

$$\int_0^4 (|x| + |x-2| + |x-4|)\,dx.$$

Solution. By *definition*,

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$$|x-2| = \begin{cases} x-2, & \text{if } x-2 \geq 0 \text{ i.e. if } x \geq 2 \\ -(x-2), & \text{if } x-2 < 0, \text{ i.e. if } x < 2 \end{cases}$$

and $|x-4| = \begin{cases} x-4, & \text{if } x-4 \geq 0 \text{ i.e. if } x \geq 4 \\ -(x-4), & \text{if } x-4 < 0, \text{ i.e. if } x < 4. \end{cases}$

Now, $\int_0^4 (|x| + |x-2| + |x-4|)\,dx$

$$= \int_0^2 (|x| + |x-2| + |x-4|)\,dx + \int_2^4 (|x| + |x-2| + |x-4|)\,dx$$

$$= \int_0^2 [x + \{-(x-2)\} - (x-4)]\,dx + \int_2^4 [x + (x-2) - (x-4)]\,dx$$

$$= \int_0^2 (-x+6)dx + \int_2^4 (x+2)dx$$

$$= \left[-\frac{x^2}{2} + 6x\right]_0^2 + \left[\frac{x^2}{2} + 2x\right]_2^4$$

$$= \left[\left(-\frac{4}{2}+12\right) - (-0+0)\right] + \left[\left(\frac{16}{2}+8\right) - \left(\frac{4}{2}+4\right)\right]$$

= 10 + (16 – 6) = 10 + 10 = 20.

Q. 29. Solve the following differential equation :

$2y\,e^{x/y}\,dx + (y - 2x\,e^{x/y})\,dy = 0$.

Solution. The given equation is :

$$2y\,e^{x/y}\,dx + (y - 2x\,e^{x/y})\,dy = 0$$

$$\Rightarrow \quad 2ye^{x/y}\,dx = (2xe^{x/y} - y)\,dx$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}} \qquad ....(1)$$

Put x = vy so that $\frac{dx}{dy} = v + y\frac{dv}{dy}$.

$$\therefore (1) \text{ becomes : } v + y\frac{dv}{dy} = \frac{2vye^v - y}{2ye^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{2vye^v - y}{2ya^v} - v$$

$$= \frac{2vye^v - y - 2vye^v}{2ye^v}$$

$$\Rightarrow \quad = \frac{-y}{2ye^v}$$

$$\Rightarrow \quad \frac{dv}{dy} = \frac{-1}{2ye^v}$$

$$\Rightarrow \quad 2e^v\,dv = -\frac{1}{y}dy \qquad | \textit{ Variables Separable}$$

Integrating, $2\int e^v\,dv = -\int \frac{1}{y}dy + c$

$$\Rightarrow \quad 2e^v = -\log y + c$$

$\Rightarrow \quad 2e^{x/y} + \log y = c$, which is the reqd. solution.

Or

When an object is thrown vertically upword it is under the effect of gravity and air resistance. For small objects, the focus due to air resistance is numerically equal to some constant k times v, where v is the velocity of the object (in m/s) in time t (s). The situation can be modelled as the differential equation shown below.

$$\frac{dv}{dt} = -F_R - mg,$$

which m is the mass of the object in kg. $\frac{dx}{dt}$ is the acceleration of the object in m/s^2.

F_R is the force due to air resistance, g is the accerlation due to gravity (10 m/s^2).

A tennis ball of mass 0.050 kg. is but upwords with a velocity of 10 m/s. An air resistance merically equal to 0.400 acts on the ball.

(i) Model the above situation, using differated equation

(ii) Write an expression for the velocity of the ball in tenis of the time.

Show your words.

Solution. (*i*) Models for the situation as as below :

$$0.05\frac{dv}{dt} = -0.4v - 0.5$$

$$\Rightarrow \quad \frac{dx}{dt} + 8v = 0 \qquad ....(1) \mid \textbf{\textit{Linear Equation}}$$

(*i*) Here P = 8 and Q = 10

$$\left[\text{Composing and } \frac{dy}{dt}, \text{P}y = \text{Q}\right]$$

$$\therefore \quad \text{I.F.} = e^{\int \text{P}\,dt}$$

$$= e^{\int 8\,dt} = e^{8t}$$

Multiplying (1) by e^{8t}, we get :

$$e^{8t}\frac{dv}{dt} + 8ve^{8t} = -10e^{8t}$$

$$\Rightarrow \quad \frac{d}{dt}(ve^{8t}) = -10e^{8t}$$

Integration, $\quad \text{V}e^{8t} = -10\int e^{8t} + c$

$$\Rightarrow \quad \text{V}e^{8t} = \frac{-10}{8}e^{8t} + c$$

$$\Rightarrow \quad e = -1.25 + ce^{-2t} \qquad ....(2)$$

When V (0) = 10, them

$$10 = -1.25 + c \Rightarrow c = 11.25$$

Putting in (2), V = – 1.25 + 11.25 e^{-8t}, which is the reqd. expression.

Q. 30. Maximise Z = $x + y$,

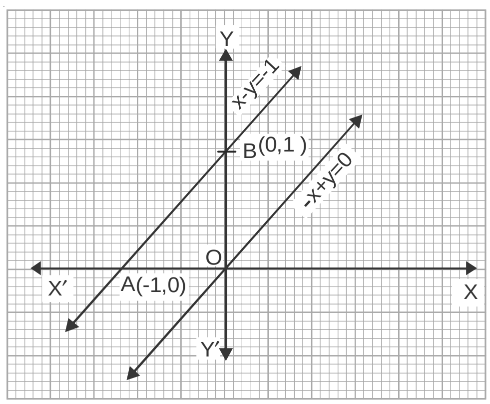
subject to $x - y \le -1, -x + y \le 0, x, y \ge 0$.

Solution : The system of constraints is :

$$x - y \le -1 \qquad ...(1)$$

$$-x + y \le 0 \qquad ...(2)$$

$$x, y \ge 0 \qquad ...(3)$$



Draw the lines $x - y = -1$ and $-x + y = 0$.

Clearly there is no feasible region.

[$\because$ *There is no common region*]

Hence, there is no maximum value of Z.

Or

The monthly incomes of Aryan and Babban are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves ₹15,000 per month, find their monthly incomes, using matrix method.

Solution. Let ₹ $3x$ and ₹ $4x$ be the monthly income of Aryan and Babban respectively.

Let ₹ $5y$ and ₹ $7y$ be the monthly expenditure of Aryan and Babban respectively.

By the question, $3x - 5y = 15000$ ...(1)

and $4x - 7y = 15000$ ...(2)

These equations can be written as AX = B ...(3),

where $A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$.

Now $|A| = \begin{vmatrix} 3 & -5 \\ 4 & -7 \end{vmatrix} = -21 + 20 = -1 \neq 0.$

$\therefore$ A is non-singular and as such A^{-1} exists.

Now adj A $= \begin{bmatrix} -7 & -4 \\ 5 & 3 \end{bmatrix}' = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$.

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-1}\begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}.$$

From (3), $A^{-1}(AX) = A^{-1}B$

$\Rightarrow \quad (A^{-1}A)X = A^{-1}B$

$\Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B$

$$\therefore \quad X = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105000 - 75000 \\ 60000 - 45000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30000 \\ 15000 \end{bmatrix} \Rightarrow x = 30{,}000.$$

Hence, monthly income of Aryan = 3(30,000) = ₹ 90,000

and monthly income of Babban = 4(30,000) = ₹ 1,20,000.

Q. 31. Minimize Z = 3x + 2y

subject to the constraints :

$x + y \geq 8$, $3x + 5y \leq 15$, $x \geq 0$, $y \geq 0$.

Solution. The system of constraints is :

$x + y \geq 8$ ...(1)

$3x + 5y \leq 15$ ...(2)

and $x, y \geq 0$ ...(3)

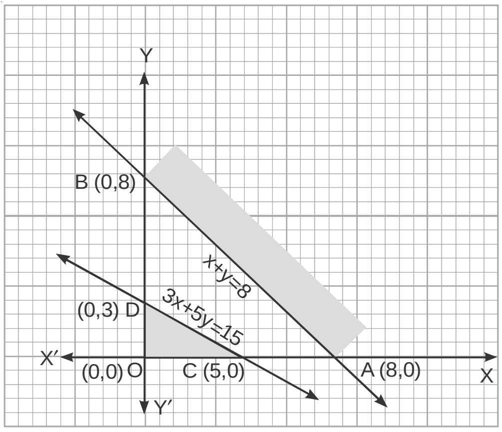


Fig.

It is observed that there is no point, which satisfies all (1) – (3) simultaneously.

Thus there is no feasible region.

Hence, there is no feasible solution.

SECTION—D

(This section comprises of long answer type questions (LAQ) of 5 marks each)

Q. 32. Find the area of the region bounded by the curve $y^2 = 4x$, y - axis and line y = 3.

Sol. $y^2 = 4x$ is right-handed parahola.

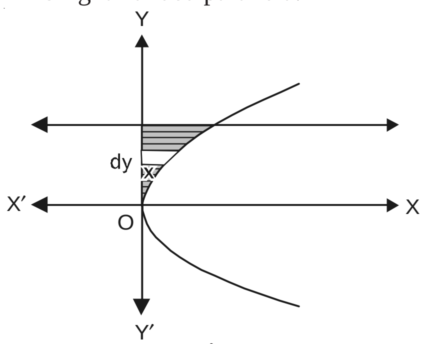


Fig.

Reqd. area $= \int_0^3 x\, dy$ **[*Taking horizonntal strips*]**

$$= \int_0^3 \frac{y^2}{4} dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3$$

$$= \frac{1}{12}[27 - 0] = \frac{27}{12} = \frac{9}{4} \text{ sq. units.}$$

Q. 33. Let L be the set of all lines in the plane and *R* be the relation in L, defined as :

$R = \{(L_1, L_2) : L_1$ is perpendicular to $L_2\}$.

Show that *R* is symmetric but neither reflexive nor transitive.

Solution. We have :

$R = \{(L_1, L_2) : L_1$ is perpendicular to $L_2\}$.

Now L_1 can't be perpendicular to itself

i.e. $(L_1, L_1) \notin R$.

Thus *R is not reflexive.*

Now $(L_1, L_2) \in R \Rightarrow L_1$ is perpendicular to L_2

$\Rightarrow L_2$ is perpendicular to L_1

$\Rightarrow (L_2, L_1) \in R$.

Thus *R is symmetric.*

Now $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$

$\Rightarrow L_1$ is perpendicular to L_2

and L_2 is perpendicular to L_3

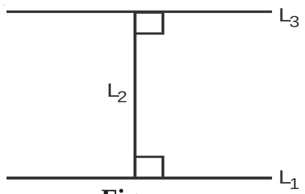


Fig.

$\Rightarrow L_1$ is parallel to L_3

$\Rightarrow L_1$ is not perpendicular to L_3

$\Rightarrow (L_1, L_3) \notin R$.

Thus *R is not transitive*.

Hence, R is symmetric but neither reflexive nor transitive.

Or

Show that the function $f: R \to R$ defined by :

$$f(x) = \frac{x}{x^2+1} \quad \forall x \in R$$

is neither one-one nor onto.

Solution. For $x_1, x_2 \in \mathbf{R}$, $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1x_2^2 + x_1 = x_1^2 x_2 + x_2$$

$$\Rightarrow x_1x_2(x_2 - x_1) = x_2 - x_1$$

$$\Rightarrow x_1 \neq x_2 \text{ or } x_1x_2 = 1.$$

Hence, there are points x_1 and x_2 with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

[For Ex. *Take* $x_1 = 2, x_2 = \frac{1}{2}$, *then* $f(x_1) = \frac{2}{4+1} = \frac{2}{5}$

and $f(x_2) = \frac{1/2}{1/4+1} = \frac{2}{5}$

Thus $f(x_1) = f(x_2) = \frac{2}{5}$ *but* $x_1 \neq x_2$]

Hence, 'f' is not one-one.

Also, 'f' is not onto.

For if so, then for $1 \in \mathbf{R}$, there exists $x \in \mathbf{R}$ such that $f(x)$ which gives $\frac{x}{x^2+1} = 1$.

But there is no such x in the domain **R**.

[$\because$ *$x^2 - x + 1 = 0$ does not give any real value of x*]

Q. 34. Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right angled triangle. Hence, find the area of the triangle.

Solution. Let $(2\hat{i} - \hat{j} + \hat{k})$, $(\hat{i} - 3\hat{j} - 5\hat{k})$ and $(3\hat{i} - 4\hat{j} - 4\hat{k})$ be the position vectors of the vertices A, B and C respectively.

$$\therefore \quad \overrightarrow{AB} = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{BC} = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}$$

and $\overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k}) = -\hat{i} + 3\hat{j} + 5\hat{k}$.

Now, $\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 3\hat{j} + 5\hat{k})$

$= (2)(-1) + (-1)(3) + (1)(5) = -2 - 3 + 5 = 0$.

Thus $\overrightarrow{BC} \perp \overrightarrow{CA} \Rightarrow \angle C = 90°$.

Hence, ΔABC is rt. angled.

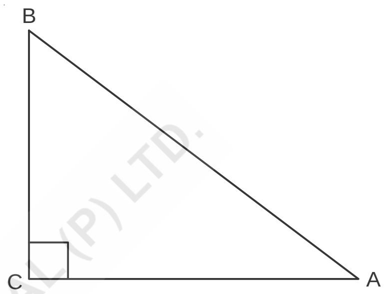


Fig.

And area of triangle $= \frac{1}{2}|\overrightarrow{CA} \times \overrightarrow{CB}|$ ...(1)

But
$$\overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & 5 \\ -2 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-3-5) - \hat{j}(1+10) + \hat{k}(-1+6)$$

$$= -8\hat{i} - 11\hat{j} + 5\hat{k}.$$

$$\therefore \quad |\overrightarrow{CA} \times \overrightarrow{CB}| = \sqrt{64+121+25} = \sqrt{210}.$$

Hence, area of triangle $= \frac{1}{2}\sqrt{210}$ sq. units

Or

Find the shortest distance between the lines:

$$\vec{\mathbf{r}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}} + \lambda(3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - \hat{\mathbf{k}})$$

and $\vec{\mathbf{r}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}} + \mu(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}})$**.**

Solution. Comparing given equations with :

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, we have :

$\vec{b}_1 = 3\hat{i} - 4\hat{j} - \hat{k}$,

$\vec{b}_2 = \hat{i} + \hat{j} + 5\hat{k}$

and $\vec{a}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$,

$\vec{a}_2 = 2\hat{i} - \hat{j} + \hat{k}$.

$$\text{Now } \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & -1 \\ 1 & 1 & 5 \end{vmatrix}$$

$$= \hat{i}(-20+1) - \hat{j}(15+1) + \hat{k}(3+4)$$

$$= -19\hat{i} - 16\hat{j} + 7\hat{k}.$$

$$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-19)^2 + (-16)^2 + (7)^2}$$

$$= \sqrt{361 + 256 + 49} = \sqrt{666}.$$

$$\text{Also, } \vec{a_2} - \vec{a_1} = (2-1)\hat{i} + (-1-2)\hat{j} + (1+3)\hat{k}$$

$$= \hat{i} - 3\hat{j} + 4\hat{k}.$$

$\therefore$ d, the shortest distance between the given lines is given by :

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \left| \frac{(-19\hat{i} - 16\hat{j} + 7\hat{k}).(\hat{i} - 3\hat{j} + 4\hat{k})}{\sqrt{666}} \right|$$

$$= \left| \frac{(-19)(1) + (-16)(-3) + (7)(4)}{\sqrt{666}} \right|$$

$$= \left| \frac{-19 + 48 + 28}{\sqrt{666}} \right| = \left| \frac{57}{\sqrt{666}} \right| = \frac{57}{666}\sqrt{666} \text{ units.}$$

Q. 35. A box contains 10 tickets, 2 of which carry a prize of ₹ 8 each, 5 of which carry a prize ₹ 4 each and remaining 3 carry a prize of ₹ 2 each . If one ticket is drawn at random, find the mean value of the prize.

Solution. Here 'X' takes values 8, 4 and 2.

$$\therefore \text{P}(8) = \frac{2}{10} = \frac{1}{5}, \text{P}(4) = \frac{5}{10} = \frac{1}{2} \text{ and}$$

$$\text{P}(2) = \frac{3}{10}.$$

$\therefore$ Probability distribution is :

X	8	4	2
P(X)	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{3}{10}$

$$\therefore \quad \mu = E(X) = \Sigma X\, P(X)$$

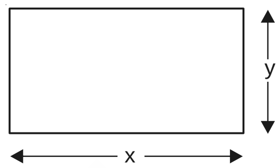
$$= 8 \times \frac{1}{5} + 4 \times \frac{1}{2} + 2 \times \frac{3}{10}$$

$$= \frac{8}{5} + 2 + \frac{3}{5} = \frac{8+10+3}{5} = \frac{21}{5}.$$

SECTION—E

(This section comprsies of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36. Manjit wants to donate a rectangular plot of land for a school in his village. When he was asked to give dimensions of the plot, he told that if its length is decreased by 50 m and breadth is increased by 50 m, then its area will remain same, but if length is decreased by 10 m and breadth is decreased by 20 m, then its area will decrease by 5300 m^2.



Based on the above information, answer the following :

Ans. (I) Find the equations in terms of x and y.

(II) Find the value of x (length of rectangular field).

(III) Find the value of y (breadth of rectangular field).

(I) $(x - 50)(y + 50) = xy$ ...(1)

$(x - 10)(y - 20) = xy - 5300$ ...(2)

Solving (1) and (2), $x = 200$, $y = 150$

$\therefore$ $x - y = 50$ and $2x + y = 550$

(II) $x = 200$ m

(III) $y = 150$ m

Or

How much is the area of rectangular field ?

Ans. Area = 200 × 150 = 30000 sq. m.

Q. 37. A class XII student appearing for a competitive examination was asked to attempt the following questions.

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non zero vectors.

Based on the above information answer the following questions :

(I) **If $\vec{a}$ and $\vec{b}$ are such that $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then prove the $\vec{a} \perp \vec{b}$.**

(II) **If $\vec{a} = \hat{i} - 2\hat{j}$, $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$, then evaluate : $(2\vec{a} + \vec{b}).[(\vec{a} + \vec{b}) \times (\vec{a} - 2\vec{b})]$.**

(III) **If $\vec{a}$ and $\vec{b}$ are unit vectors and θ be the angle between them, then find $|\vec{a} - \vec{b}|$.**

Ans. (I) $|\vec{a} \times \vec{b}| = |\vec{a} - \vec{b}|$.

$$\text{Squaring, } |\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$$

$$\Rightarrow 4\vec{a}.\vec{b} = 0 \Rightarrow \vec{a}.\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}.$$

(II) $2\vec{a}+\vec{b} = 2(\hat{i}-2\hat{j})+(2\hat{i}+\hat{j}+3\hat{k})$

$= 4\hat{i}-3\hat{j}+3\hat{k}$ ...(1)

and $(\vec{a}+\vec{b})\times(\vec{a}-2\vec{b})$

$= (3\hat{i}-\hat{j}+3\hat{k})\times(-3\hat{i}-4\hat{j}-6\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 3 \\ -3 & -4 & -6 \end{vmatrix}$$

$= \hat{i}(6+12)-\hat{j}(-18+9)+\hat{k}(-12-3)$

$= 18\hat{i}+9\hat{j}-15\hat{k}$ ...(2)

From (1) and (2), $(2\vec{a}+\vec{b}).[(\vec{a}+\vec{b})\times(\vec{a}-2\vec{b})]$

$= (4\hat{i}-3\hat{j}+3\hat{k}).(18\hat{i}+9\hat{j}-15\hat{k})$

= (4) (18) + (– 3) (9) + (3) (– 15)

= 72 – 27 – 45 = 0.

(III) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2-2\vec{a}.\vec{b}$

$= 1 + 1 - 2\ (1)\ (1) \cos\theta$

$= 2\ (1 - \cos\theta)$

$= 4\sin^2\dfrac{\theta}{2}$

$\therefore |\vec{a}-\vec{b}| = 2\sin\dfrac{\theta}{2}$.

Or

Find the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$ as diagonals.

Area $= \dfrac{1}{2}|(\vec{a}\times\vec{b})| = \dfrac{\sqrt{70}}{2}$ sq. units.

Q. 38. Senior students tend to stay up al night and therefore are not able to wake up on time in morning. Not only this but their dependence on tuitions further leads to absenteeism in school. Of the students in class XII, it is known that 30% of the students have 100% attendance. Previous year results report that 80% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the class XII.

Using above information answer the following :

(I) Find the conditional probability that a student attains A grade given that he is not 100% regular student.

(II) Find the probability of attaining A grade by the students of class XII.

(III) Find the probability that student is 100% regular given that he attains A grade.

Ans. Let the events be :

E_1 : Student is regular

E_2 : Student is irregular

and A : Student attains grade A.

$\therefore$ $P(E_1) = \dfrac{30}{100}$, $P(E_2) = \dfrac{70}{100}$

and $P(A/E_1) = \dfrac{80}{100}$ and $P(A/E_2) = \dfrac{10}{100}$.

(I) Required probability $= P(A/E_2) = \dfrac{10}{100} = \dfrac{1}{10}$.

(II) Required probility = P(A)

$= P(E_1)\ P(A/E_1) + P(E_2)\ P(A/E_2)$

$= \dfrac{30}{100}\times\dfrac{80}{100}+\dfrac{70}{100}\times\dfrac{10}{100}$

$= \dfrac{31}{100}$.

(III) Required Probility $= P(E_1/A)$

$$= \frac{P(E_1)P(A/E_1)}{P(E_1)\ P(A/E_1) + P(E_2)\ P(A/E_2)}$$

[*Boyes' Theorem*]

$$= \frac{\frac{30}{100}\times\frac{80}{100}}{\frac{30}{100}\times\frac{80}{100}+\frac{70}{100}\times\frac{10}{100}} = \frac{24}{31}.$$

Or

Find the probability that student is irregular given that he attains A grade.

Required probability $= P(E_2/A)$

$$= \frac{P(E_2)\ P(A/E_2)}{P(E_1)\ P(A/E_1) + P(E_2)\ P(A/E_2)}$$

[*Boyes' Theorem*]

$$= \frac{\frac{70}{100}\times\frac{10}{100}}{\frac{30}{100}\times\frac{80}{100}+\frac{70}{100}\times\frac{10}{100}} = \frac{7}{31}.$$

Holy Faith New Style Sample Paper–5

(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions : Same as Holy Faith New Style Sample Paper–1.

SECTION—A

(Multiple choice questions, each question carries 1 mark)

1. A relation R in the set A = {1, 2, 3} is defined as :
R = {(1, 1), (1, 2), (2, 2), (3, 3)}.
Which of the following ordered pair in R shall be removed to make it an equivalence relation in A ?

(*a*) (1, 1) (*b*) (1, 2)
(*c*) (2, 2) (*d*) (3, 3).

Ans. (*d*) (3, 3).

2. The principal value of $[\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})]$ is :

(*a*) π (*b*) $-\frac{\pi}{2}$
(*c*) 0 (*d*) $2\sqrt{3}$.

Ans. (*b*) $-\frac{\pi}{2}$

3. If A = $\begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix}$ and A = A^T, where A^T is the transpose of the matrix A, then :

(*a*) $x = 0, y = 5$ (*b*) $x = y$
(*c*) $x + y = 5$ (*d*) $x = 5, y = 0$.

Ans. (*b*) $x = y$

4. For any 2 × 2 matrix P, which of the following matrices can be Q such that PA = QP ?

(*a*) [1] (*b*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(*c*) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(*d*) No such matrix exists as matrix multiplication is not commutative.

Ans. (*d*) No such matrix exists as matrix multiplication is not commutative.

5. If the area of the triangle with vertices (–3, 0), (3, 0) and (0, *k*) is 9 sq. units, then the value/s of '*k*' will be :

(*a*) 9 (*b*) ± 3
(*c*) –9 (*d*) 6.

Ans. (*b*) ± 3

6. If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to :

(*a*) $\frac{4x^3}{1-x^4}$ (*b*) $\frac{-4x}{1-x^4}$

(*c*) $\frac{1}{4-x^4}$ (*d*) $\frac{-4x^3}{1-x^4}$.

Ans. (*b*) $\frac{-4x}{1-x^4}$.

7. If $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then $\frac{du}{dv}$ is :

(*a*) $\frac{1}{2}$ (*b*) x

(*c*) $\frac{1-x^2}{1+x^2}$ (*d*) 1.

Ans. (*d*) 1.

8. The function $f(x) = x^3 + 3x$ is increasing in the interval:

(*a*) $(-\infty, 0)$ (*b*) $(0, \infty)$
(*c*) **R** (*d*) **R**.

Ans. (*b*) $(0, \infty)$

9. The antiderivative of $\left(\frac{x^3 + 5x^2 - 4}{x^2}\right)$ is equal to :

(*a*) $\frac{x^3}{3} + \frac{5x^2}{2} - \frac{4}{x^2} + c$

(*b*) $\frac{x^2}{2} + 5x + \frac{4}{x} + c$

(*c*) $\frac{x^3}{3} + 5x - \frac{4}{x} + c$

(*d*) $\frac{x^4}{4} - \frac{5x^3}{3} - 4x + c$.

Ans. (*b*) $\frac{x^2}{2} + 5x + \frac{4}{x} + c$

10. $\int_0^{\pi/6} \sec^2\left(x - \frac{\pi}{6}\right)dx$ is equal to:

(*a*) $\frac{1}{\sqrt{3}}$ (*b*) $-\frac{1}{\sqrt{3}}$

(*c*) $\sqrt{3}$ (*d*) $-\sqrt{3}$.

Ans. (*a*) $\frac{1}{\sqrt{3}}$

11. Area bounded by the curve $y = x^3$, the *x*-axis and the ordinates $x = -2$ and $x = 1$ is :

(*a*) – 9 (*b*) $-\frac{15}{4}$

(*c*) $\frac{15}{4}$ (*d*) $\frac{17}{4}$.

Ans. (*d*) $\frac{17}{4}$

12. The number of arbitrary constants in the particular solution of a differential equation of second order is (are):

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3.

Ans. (*a*) 0

13. Which of the following differential equations has $y = x$ as one of its particular solutions ?

(*a*) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$

(*b*) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = x$

(*c*) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$

(*d*) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$.

Ans. (*c*) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$

14. Two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if :

(*a*) $a_1b_1 + a_2b_2 + a_3b_3 = 0$

(*b*) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

(*c*) $a_1 = b_1, a_2 = b_2, a_3 = b_3$

(*d*) $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$.

Ans. (*b*) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

15. If $\vec{a} + \vec{b} = \hat{i}$ and $\vec{a} = 2\hat{i} - 2\hat{j} + 2\hat{k}$, then $|\vec{b}|$ equals :

(*a*) $\sqrt{14}$ (*b*) 3

(*c*) $\sqrt{12}$ (*d*) $\sqrt{17}$.

Ans. (*b*) 3

16. The angle between the lines :

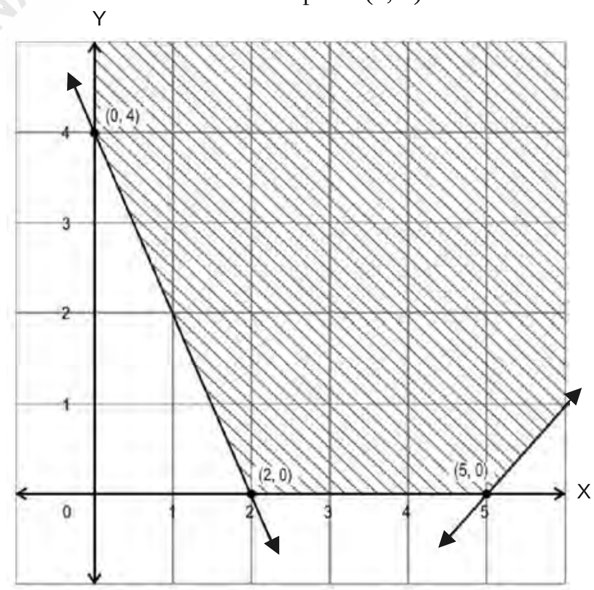
$2x = 3y = -z$ and $6x = -y = -4z$ is :

(*a*) 0° (*b*) 30°

(*c*) 45° (*d*) 90°.

Ans. (*d*) 90°.

17. A linear programming problem (LPP) along with the graph of its constraints is shown below. The corresponding objective function is Minimize : $Z = 3x + 2y$. The minimum value of the objective function is obtained at the corner point (2, 0).



The optimal solution of the above linear programming problem ....

(*a*) does not exist as the feasible region is unbounded

(*b*) does not exist as the inequality $3x + 2y < 6$ does not have any point in common with the feasible region

(*c*) exists as the inequality $3x + 2y > 6$ has infinitely many points in common with the feasible region

(*d*) exists as the inequality $3x + 2y < 6$ does not have any point in common with the feasible region.

Ans. (*a*) does not exist as the feasible region is unbounded

18. Ashima can hit a target 2 out of 3 times. She tried to hit the target twice. The probability that she missed the target exactly once is :

(*a*) $\frac{2}{3}$ (*b*) $\frac{1}{3}$

(*c*) $\frac{4}{9}$ (*d*) $\frac{1}{9}$.

Ans. (*c*) $\frac{4}{9}$

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.
(*b*) Both 'A' and 'R' are true but 'R' is not the correct explanation of 'A'.
(*c*) 'A' is true but 'R' is false.
(*d*) 'A' is false but 'R' is true.

Q. 19. Assertion (A) : Area enclosed by the curve $y = e^{x^5}$ between the lines $x = p$ and $x = q$ and x-axis is $\int_p^q e^{x^5}\, dx$.

Reason (R) : e^{x^2} is an increasing function.

Ans. (*b*) Here $y = e^{x^5}$.

$\therefore \frac{dy}{dx} = e^{x^5}.5x^4 > 0 \Rightarrow y$ is an increasing function.

And area enclosed = $\int_p^q e^{x^5} dx$.

Q. 20. Assertion (A) : Shortest distance between two skew-lines :

$$\frac{x+3}{-4} = \frac{y-6}{3} = \frac{z}{2} \text{ and } \frac{x+2}{-4} = \frac{y}{1} = \frac{z-7}{8} \text{ is 9.}$$

Reason (R) : Two lines are skew if there exists no plane passing through them

Ans. (*b*) Using usual formula, S.D. = 9.
And statement Reason (R) holds.

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Find the principal value of cosec $^{-1}$ (2).

Ans. Let cosec $^{-1}$ (2) = y, where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\Rightarrow$ cosec $y = 2 = \text{cosec}\, \frac{\pi}{6} \Rightarrow y = \frac{\pi}{6}$.

Hence, the reqd. principal value is $\frac{\pi}{6}$

Or

Find the value of :

$$\sin^{-1}\left(2 \tan^{-1}\frac{1}{4}\right) + \cos(\tan^{-1} 2\sqrt{2}).$$

Solution. To evaluate : $\sin^{-1}\left(2\tan^{-1}\frac{1}{4}\right)$.

Put $\tan^{-1}\frac{1}{4} = \theta$ so that $\tan\theta = \frac{1}{4}$.

Now, $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\left(\frac{1}{4}\right)}{1+\left(\frac{1}{4}\right)^2} = \frac{1/2}{1+1/16}$

$= \frac{1/2}{17/16} = \frac{8}{17}$ ...(1)

To evaluate : $\cos(\tan^{-1} 2\sqrt{2})$.

Put $\tan^{-1} 2\sqrt{2} = \phi$ so that $\tan\phi = 2\sqrt{2}$.

$\therefore \cos\phi = \frac{1}{3}$ ...(2)

Hence, $\sin^{-1}\left(2\tan^{-1}\frac{1}{4}\right) + \cos(\tan^{-1} 2\sqrt{2})$

$= \frac{8}{17} + \frac{1}{3}$ [*Using (1) & (2)*]

$= \frac{24+17}{51} = \frac{41}{51}$.

Q. 22. For the curve y = 5x – 2x^3, if x increases at the rate of 2 units/sec., find the rate of change of the slope of the curve when x = 3.

Solution. The given curve is $y = 5x - 2x^3$.

$\therefore \frac{dy}{dx} = 5 - 6x^2$

i.e. $m = 5 - 6x^2$, where 'm' is the slope.

$\therefore \frac{dm}{dt} = -12x\frac{dx}{dt} = -12x(2) = -24x$.

$\therefore \left.\frac{dm}{dt}\right]_{x=3} = -24(3) = -72$.

Hence, the rate of the change of the slope = – 72.

Q. 23. It is given that at x = 1, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of 'a'.

Solution : We have : $f(x) = x^4 - 62x^2 + ax + 9$.

$\therefore f'(x) = 4x^3 - 124x + a$

and $f''(x) = 12x^2 - 124$.

Now $f'(1) = 0$ [*Given*]

$\Rightarrow 4\,(1) - 124\,(1) + a = 0$

$\Rightarrow \quad a = 120.$

Now $f''(1) = 12\,(1) - 124 = -112 < 0.$

Thus $f(x)$ attains its maximum value at $x = 1$, where $a = 120$.

Hence, $a = 120$.

Or

Sum of two numbers is 5. If the sum of the cubes of these numbers is least, then find the sum of the squares of these numbers.

Solution. Let 'x' and 'y' be two numbers.

By the question, $x + y = 5$ ... (1)

Let $S = x^3 + y^3$

$= x^3 + (5 - x)^3$ ***[Using (1)]***

$= x^3 + (125 - 75x + 15x^2 - x^3)$

$= 15x^2 - 75x + 125.$

$$\therefore \quad \frac{dS}{dx} = 30x - 75.$$

For S to be least, $\frac{dS}{dx} = 0$

$\Rightarrow \quad 30x - 75 = 0$

$\Rightarrow \quad x = \frac{5}{2}.$

Now $\frac{d^2S}{dx^2} = 30,$

which is positive for $x = \frac{5}{2}$.

Hence, the least sum of squares of the numbers

$$= \left(\frac{5}{2}\right)^2 + \left(5 - \frac{5}{2}\right)^2 = \frac{25}{4} + \frac{25}{4} = \frac{25}{2}.$$

Q. 24. Evaluate : $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx.$

Solution : Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx.$

Put x = tan θ so that $dx = \sec^2\theta\, d\theta$.

When $x = 0$, $\tan\theta = 0 \Rightarrow \theta = 0$.

When $x = 1$, $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$.

$$\therefore \; I = \int_0^{\pi/4} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right).\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} 2\theta\sec^2\theta\, d\theta = 2\int_0^{\pi/4} \theta\sec^2\theta\, d\theta$$

$$= 2\left[\theta\tan\theta - \int (1)\tan\theta\, d\theta\right]_0^{\pi/4}$$

[*Integrating by parts*]

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right]_0^{\pi/4}$$

$$= 2\left[\left(\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right|\right) - (0 + \log|1|)\right]$$

$$= 2\left[\frac{\pi}{4}(1) + \log\left(\frac{1}{\sqrt{2}}\right) - 0\right]$$

$$= 2\left[\frac{\pi}{4} + \log 1 - \frac{1}{2}\log 2\right] = 2\left[\frac{\pi}{4} + 0 - \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2.$$

Q. 25. Find $\frac{dy}{dx}$, when $\sin^2 x + \cos^2 y = 1$.

Solution. We have : $\sin^2 x + \cos^2 y = 1.$

Diff. w.r.t. x, $2\sin x.\frac{d}{dx}(\sin x) + 2\cos y.\frac{d}{dx}(\cos y) = 0$

$\Rightarrow 2\sin x\cos x + 2\cos y(-\sin y)\frac{dy}{dx} = 0$

$\Rightarrow \sin 2x - \sin 2y\frac{dy}{dx} = 0.$

Hence, $\frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}.$

SECTION—C

(This section comparies of short answer type questions (SAQ) of 3 marks each)

Q. 26. Find : $\int \frac{x+1}{(x^2+1)\,x}\,dx.$

Solution. Let $\frac{x+1}{(x^2+1)\,x} \equiv \frac{Ax+B}{x^2+1} + \frac{C}{x}$ ....(1)

Multiplying by $x(x^2 + 1)$, we get :

$x + 1 \equiv (Ax + B)\,x + C\,(x^2 + 1).$

Putting $x = 0$, $1 = C\,(0 + 1) \Rightarrow C = 1.$

Equating coefficients of x, $1 = B \Rightarrow B = 1$.

Equating coefficients of x^2, $0 = A + C \Rightarrow A = -C \Rightarrow A = -1$.

$\therefore$ From (1), $\dfrac{x+1}{(x^2+1)\,x} = \dfrac{-x+1}{x^2+1} + \dfrac{1}{x}$.

$$\therefore \int \frac{x+1}{(x^2+1)\,x}dx = \int \frac{-x+1}{x^2+1}dx + \int \frac{1}{x}dx$$

$$= -\frac{1}{2}\int \frac{2x-2}{x^2+1}dx + \int \frac{1}{x}dx$$

$$= -\frac{1}{2}\int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx + \int \frac{1}{x}dx$$

$$= -\frac{1}{2}\log\left|x^2+1\right| + \tan^{-1}x + \log|x| + c$$

$$= -\frac{1}{2}\log(x^2+1) + \tan^{-1}x + \log|x| + c.$$

[$\because x^2 \geq 0 \Rightarrow x^2 + 1 > 0 \Rightarrow |x^2 + 1| = x^2 + 1$]

Example 27. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance?

Solution. Let the events E_1, E_2 and A be as below :

E_1 : Students having 100% attendance

E_2 : Students which are irregular

and A : Students getting A-grade.

We have $P(E_1) = \dfrac{30}{100} = \dfrac{3}{10}$ and $P(E_2) = \dfrac{70}{100} = \dfrac{7}{10}$.

Also, $P(A/E_1) = \dfrac{70}{100} = \dfrac{7}{10}$ and $P(A/E_2) = \dfrac{10}{100} = \dfrac{1}{10}$.

By *Bayes' Theorem,*

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{3}{10}\times\frac{7}{10}}{\frac{3}{10}\times\frac{7}{10}+\frac{7}{10}\times\frac{1}{10}}$$

$$= \frac{21}{21+7} = \frac{21}{28} = \frac{3}{4}.$$

Example 28. Find : $\displaystyle\int e^x \frac{\sqrt{1+\sin 2x}}{1+\cos 2x}dx.$

Solution. Let $I = \displaystyle\int e^x \frac{\sqrt{1+\sin 2x}}{1+\cos 2x}dx$

$$= \int e^x \frac{\sqrt{\cos^2 x + \sin^2 x + 2\cos x \sin x}}{1+\cos 2x}dx$$

$$= \int e^x \frac{\sqrt{(\cos x + \sin x)^2}}{2\cos^2 x}dx$$

$$= \frac{1}{2}\int e^x\left(\frac{\cos x + \sin x}{\cos^2 x}\right)dx$$

$$= \frac{1}{2}\int e^x\left(\frac{1}{\cos x} + \frac{1}{\cos x}.\frac{\sin x}{\cos x}\right)dx$$

$$= \frac{1}{2}\int e^x(\sec x + \sec x\tan x)dx$$

I"Form : $\int e^x(f(x) + f'(x))\,dx$"

$$= \frac{1}{2}\left[\int \sec x . e^x\,dx + \int \sec x\tan x . e^x dx\right]$$

$$= \frac{1}{2}\left[\sec x . e^x - \int \sec x \tan x . e^x dx + \int \sec x \tan x . e^x dx\right]$$

[*Integrating first integral by parts*]

$$= \frac{1}{2}[\sec x . e^x] + c$$

$$= \frac{1}{2}e^x \sec x + c.$$

Or

Evaluate : $\displaystyle\int_0^\pi \log(1+\cos x)\,dx.$

Solution. $I = \displaystyle\int_0^\pi \log(1+\cos x)\,dx$ ...(1)

$\therefore$ $I = \displaystyle\int_0^\pi \log[1+\cos(\pi - x)]\,dx$ [*Property V*]

$\Rightarrow$ $I = \displaystyle\int_0^\pi \log(1-\cos x)\,dx$ ...(2)

Adding (1) and (2), we get :

$$2I = \int_0^{\pi} [\log(1+\cos x)(1-\cos x)]\,dx$$

$$= \int_0^{\pi} \log(1-\cos^2 x)\,dx$$

$$= \int_0^{\pi} \log \sin^2 x\,dx$$

$$= 2\int_0^{\pi} \log \sin x\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \log \sin x\,dx$$

$$= 2\int_0^{\pi/2} \log \sin x\,dx \qquad ...(3)$$

[$\because$ *log (sin (π – x) = log sin x*]

$$= 2\int_0^{\pi/2} \log \sin\left(\frac{\pi}{2}-x\right)dx$$

[*Property V*]

$$= 2\int_0^{\pi/2} \log \cos x\,dx \qquad ...(4)$$

Adding (3) and (4),

$$2I = 2\int_0^{\pi/2} \log \sin x \cos x\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx - \log 2\int_0^{\pi/2} 1.dx$$

$$= I_1 - \log 2\,[x]_0^{\pi/2} = I_1 - \frac{\pi}{2}\log 2 \qquad ....(5)$$

Now $\quad I_1 = \int_0^{\pi/2} \log \sin 2x\,dx.$

Put 2x = t so that 2 dx = dt *i.e.* $dx = \frac{1}{2}dt$.

When $x = 0$, $t = 0$, when $x = \frac{\pi}{2}$, $t = \pi$.

$$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi} \log \sin t\,dt = \frac{1}{2}\int_0^{\pi} \log \sin x\,dx$$

$$= \frac{1}{2}\times 2\int_0^{\pi/2} \log \sin x\,dx$$

$$\Rightarrow \quad I_1 = \frac{1}{2}I.$$

From (5), $\quad I = \frac{1}{2}I - \frac{\pi}{2}\log 2$

$$\Rightarrow \quad \frac{1}{2}I = -\frac{\pi}{2}\log 2.$$

Hence, $\quad I = -\pi \log 2.$

Q. 29. Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of the differential equation :

$(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$, where c is a parameter.

Solution. We have : $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)dy$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \qquad | \textit{ Homogeneous}$$

Put y = vx so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$$\therefore \quad v + x\frac{dv}{dx} = \frac{x^3 - 3v^2x^3}{v^3x^3 - 3vx^3}$$

$$= \frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v} - v$$

$$= \frac{1-3v^2-v^4+3v^2}{v^3-3v} = \frac{1-v^4}{v(v^2-3)}$$

$$\Rightarrow \quad \frac{v(v^2-3)}{1-v^4}dv = \frac{dx}{x} \qquad | \textit{ Variables Separable}$$

$$\text{Integrating,} \int \frac{(v^3-3v)}{(1-v^2)(1+v^2)}dv = \int \frac{dx}{x} \qquad ...(1)$$

$$\text{Now} \quad \frac{v^3-3v}{(1-v^2)(1+v^2)} = \frac{v^3-3v}{(1-v)(1+v)(1+v^2)}$$

$$\equiv \frac{A}{1-v} + \frac{B}{1+v} + \frac{Cv+D}{1+v^2} \qquad ...(2)$$

$$\therefore \quad v^3 - 3v \equiv A(1+v)(1+v^2) + B(1-v)(1+v^2) + (Cv+D)(1-v^2)$$

Putting $v = 1$, $\quad 1 - 3 = A(2)(2) \quad \Rightarrow \quad A = -\frac{1}{2}$.

Putting $v = -1$, $\quad -1 + 3 = B(2)(1+1) \quad \Rightarrow \quad B = \frac{1}{2}$.

Putting $v = 0$, $\quad 0 = A(1)(1) + B(1)(1) + D(1)$

$\Rightarrow \quad 0 = -\frac{1}{2} + \frac{1}{2} + D \quad \Rightarrow \quad D = 0.$

Comparing coeff. of v^3, $1 = A - B - C$

$\Rightarrow \quad C = A - B - 1 = -\frac{1}{2} - \frac{1}{2} - 1 = -2.$

$\therefore$ From (2), $\frac{v^3 - 3v}{(1-v^2)(1+v^2)}$

$$= \frac{-1/2}{1-v} + \frac{1/2}{1+v} + \frac{-2v}{1+v^2}.$$

$$\therefore \int \frac{v^3 - 3v}{(1-v^2)(1+v^2)} dv = \frac{1}{2}\log|1-v| + \frac{1}{2}\log|1+v| - \log|1+v^2|$$

$$= \frac{1}{2}\log|1-v^2| - \log|1+v^2|.$$

From (1), $\frac{1}{2}\log\left|1-v^2\right| - \log|1+v^2| = \log|x| + \log|c'|$

$$\Rightarrow \quad \log\left|\frac{\sqrt{1-v^2}}{1+v^2}\right| = \log|x| + \log|c'|$$

$$\Rightarrow \quad \log\left|\frac{\sqrt{1-\frac{y^2}{x^2}}}{1+\frac{y^2}{x^2}}\right| = \log|x| + \log|c'|$$

$$\Rightarrow \quad \log\left|\frac{x\sqrt{x^2-y^2}}{x^2+y^2}\right| = \log|x| + \log|c'|$$

$$\Rightarrow \quad \frac{x\sqrt{x^2-y^2}}{x^2+y^2} = c'x$$

$$\Rightarrow \quad x\sqrt{x^2-y^2} = c'x(x^2+y^2)$$

$$\Rightarrow \quad \sqrt{x^2-y^2} = c'(x^2+y^2).$$

Squaring, $\quad x^2 - y^2 = c'^2 (x^2+y^2)^2$

$\Rightarrow \quad x^2 - y^2 = c(x^2+y^2)^2$, where $c'^2 = c$,

which is the reqd. solution.

Or

Find the particular solution of the differential equation: $\frac{dy}{dx} + \cot x . y = \cos^2 x$, given that $x = \frac{\pi}{2}, y = 0$.

Solution. The given differential equation is :

$$\frac{dy}{dx} + \cot x . y = \cos^2 x \qquad | \textit{ Linear Equation}$$

Comparing with $\frac{dy}{dx} + Py = Q$, we have :

$$P = \cot x \text{ and } Q = \cos^2 x.$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \cot x \, dx}$$

$$= e^{\log|\sin x|} = \sin x.$$

The solution is $y . \sin x = \int \cos^2 x . \sin x \, dx + c$

$$\Rightarrow \quad y . \sin x = -\int \cos^2 x(-\sin x)\, dx + c$$

$$\Rightarrow \quad y . \sin x = -\frac{\cos^3 x}{3} + c \qquad ...(1)$$

When $\quad x = \frac{\pi}{2}, y = 0$

$\therefore \quad 0 = -0 + c$

$\Rightarrow \quad c = 0.$

Putting in (1), $y \sin x = -\frac{1}{3}\cos^3 x$,

which is the reqd. particular solution.

Q. 30. Minimise Z = $x + 2y$

subject to $2x + y \geq 3$, $x + 2y \geq 6$, $x, y \geq 0$.

Show that the minimum of Z occurs at more than two points.

Solution : The system of constraints is :

$2x + y \geq 3$ ...(1)

$x + 2y \geq 6$ ...(2)

and $x \geq 0, y \geq 0$ ...(3)

The shaded region in the following figure is the feasible region determined by the system of constraints (1) – (3) :

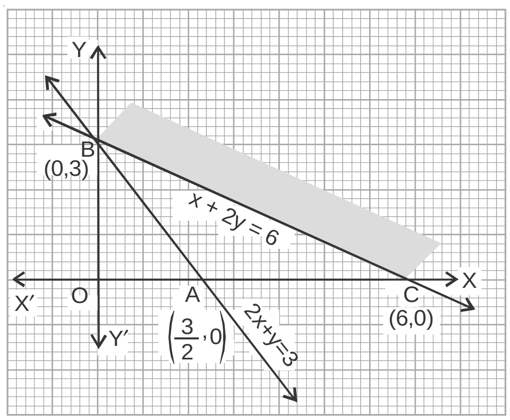


Fig.

It is observed that the feasible region is unbounded.

Applying ***Corner Point Method,*** we evaluate Z = $x + 2y$ at the corner points C (6, 0) and B (0, 3).

Corner Point	Corresponding value of Z
C : (6, 0)	**6** Minimum
B : (0, 3)	**6** Minimum

Hence, $Z_{min} = 6$ at all points (more than two points) on the line segment [CB] joining the points C (6, 0) and B (0, 3).

Or

Use product :

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

to solve the system of equations :
x + 3z = 9, – x + 2y – 2z = 4, 2x – 3y + 4z = – 3.

Solution. Here $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ -0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}' = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix} \quad ...(1)$$

The given system of equations is :

$$x + 3z = 9$$
$$-x + 2y - 2z = 4$$
$$2x - 3y + 4z = -3.$$

The system of equations can be written as AX = B...(2),

where $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$.

From (2), AX = B ⇒ $A^{-1}(AX) = A^{-1}B$
⇒ $(A^{-1}A)X = A^{-1}B$ ⇒ $IX = A^{-1}B$ ⇒ $X = A^{-1}B$.

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix} \text{[Using (1)]}$$

$$= \begin{bmatrix} -18+36-18 \\ 0+8-3 \\ 9-12+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}.$$

Hence, $x = 0$, $y = 5$ and $z = 3$.

Q. 31. Show that the points A(1, 2, 7), B(2, 6, 3) and C (3, 10, –1) are collinear.

Solution : If O be the origin,

then $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 7\hat{k}$,

$\overrightarrow{OB} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

and $\overrightarrow{OC} = 3\hat{i} + 10\hat{j} - \hat{k}$.

$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (2\hat{i} + 6\hat{j} + 3\hat{k}) - (\hat{i} + 2\hat{j} + 7\hat{k})$

$= \hat{i} + 4\hat{j} - 4\hat{k}$,

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (3\hat{i} + 10\hat{j} - \hat{k}) - (2\hat{i} + 6\hat{j} + 3\hat{k})$

$= \hat{i} + 4\hat{j} - 4\hat{k}$

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (3\hat{i} + 10\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 7\hat{k})$

$= 2\hat{i} + 8\hat{j} - 8\hat{k}$

$= 2(\hat{i} + 4\hat{j} - 4\hat{k})$.

$\therefore \quad AB = |\overrightarrow{AB}| = \sqrt{1^2 + 4^2 + (-4)^2}$

$= \sqrt{1+16+16} = \sqrt{33}$,

$BC = |\overrightarrow{BC}| = \sqrt{1^2 + 4^2 + (-4)^2}$

$= \sqrt{1+16+16} = \sqrt{33}$

and $AC = |\overrightarrow{AC}| = \sqrt{2^2 + 8^2 + (-8)^2}$

$= \sqrt{4+64+64}$

$= \sqrt{132}$

$= 2\sqrt{33}$.

Clearly AB + BC = AC.

Hence, A, B, C are collinear.

SECTION—D

(This section comparies of Long Answer Type questions (LAQ) of 5 marks each)

Q. 32. Find the area enclosed by the ellipse :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b.$$

Solution. The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

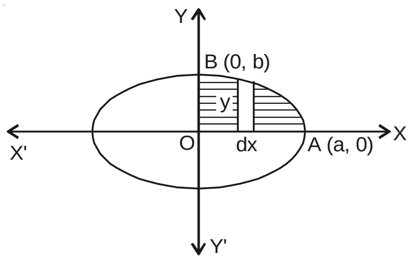


Fig.

Area of the ellipse = 4 (area of the region OABO, bounded by the curve, x-axis and the ordinates $x = 0$, $x = a$)

[$\because$ *Ellipse is symmetrical about both the axes*]

$$= 4\int_0^a y\,dx \qquad \text{[Taking vertical strips]}$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

[$\because$ (1) $\Rightarrow$ $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$. *But region OABO lies in 1st quadrant,* $\therefore$ *y is +ve*]

$$=\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=\frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{4b}{a}\left[\left\{\frac{a}{2}(0)+\frac{a^2}{2}\sin^{-1}(1)\right\}-\{0+0\}\right]$$

$$=\frac{4b}{a}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right]=\pi ab \text{ sq. units.}$$

Q. 33. Let T be the set of all triangles in a plane with *R*, a relation in T given by :

$R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$.

Show that *R* is an equivalence relation.

Solution. We have :

$R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$.

Now $(T_1, T_1) \in R$.

[$\because$ *Every triangle is congruent to itself*]

Thus *R is reflexive.*

$(T_1, T_2) \in R \Rightarrow T_1$ is congruent to T_2

$\Rightarrow T_2$ is congruent to T_1

$\Rightarrow (T_2, T_1) \in R$.

Thus *R is symmetric.*

$(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

$\Rightarrow T_1$ is congruent to T_2 and T_2 is congruent to T_3

$\Rightarrow T_1$ is congruent to T_3

$\Rightarrow (T_1, T_3) \in R$.

Thus *R is transitive.*

Hence, R is an equivalence relation.

Or

Show that the function $f : R \to R$ defined by $f(x) = x^2$ is neither one–one nor onto.

Solution.

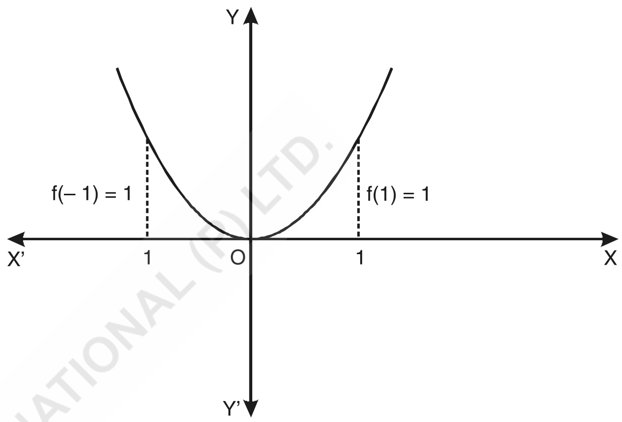


Fig.

Here, $f(-1) = f(1) = 1$ but $-1 \neq 1$.

$\therefore$ 'f' is not one–one.

Also, – 2 is in the co-domain **R** but is not the image of any element 'x' in the domain **R.**

[$\because$ *– 2 is not the square of any real number*]

$\therefore$ 'f' is not onto.

Hence, 'f' is neither one–one nor onto.

Q. 34. (a) Let $\vec{a}, \vec{b}, \vec{c}$ represent the vectors $\overrightarrow{BC}, \overrightarrow{CA}$ and $\overrightarrow{AB}$ respectively. Show that :

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$$

and deduce the rule of sines of the triangle.

(b) If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$,

show that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

Solution. (a) (i) Here $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$ and $\overrightarrow{AB}=\vec{c}$.

Then $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. [$\because$ $\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}=\vec{0}$]

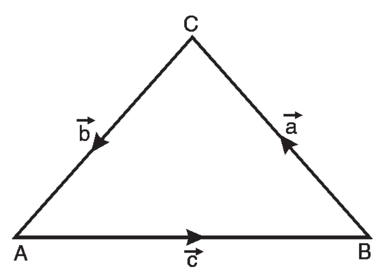


Fig.

Consider $\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{0}$

$\Rightarrow \quad \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$

$\Rightarrow \quad \vec{0} + \vec{a} \times \vec{b} - \vec{c} \times \vec{a} = 0$

$\Rightarrow \quad \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ...(1)

Similarly, $\vec{b} \times \vec{c} = \vec{a} \times \vec{b}$ ...(2)

Combining (1) and (2),

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$$

(*ii*) From (1), $|\vec{a} \times \vec{b}| = |\vec{c} \times \vec{a}|$

$\Rightarrow ab \sin C = ca \sin B \Rightarrow b \sin C = c \sin B$

$\Rightarrow \quad \dfrac{b}{\sin B} = \dfrac{c}{\sin C}.$

Similarly, $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}.$

Combining, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$,

which is the *Sine Rule.*

Another Form. *Using the vector method, prove that :*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

(*b*) This is a part of part (*a*).

Or

Find the distance between the lines L_1 and L_2 given by :

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \text{ and}$$

$$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k}).$$

Solution. Clearly, L_1 and L_2 are parallel.

Comparing given equations with :

$\vec{r} = \vec{a}_1 + \lambda \vec{b}$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}$, we have :

$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, so that

$|\vec{b}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

and $\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}$.

Now $\vec{a}_2 - \vec{a}_1 = (3-1)\hat{i} + (3-2)\hat{j} + (-5+4)\hat{k}$

$= 2\hat{i} + \hat{j} - \hat{k}$.

$\therefore \vec{b} \times (\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$

$= \hat{i}(-3-6) - \hat{j}(-2-12) + \hat{k}(2-6)$

$= -9\hat{i} + 14\hat{j} - 4\hat{k}.$

$\therefore d$, the distance between the given lines is given by :

$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

$$= \left| \frac{-9\hat{i} + 14\hat{j} - 4\hat{k}}{7} \right|$$

$$= \frac{1}{7} |-9\hat{i} + 14\hat{j} - 4\hat{k}|$$

$$= \frac{1}{7}\sqrt{81 + 196 + 16}$$

$$= \frac{1}{7}\sqrt{293} \text{ units.}$$

Q. 35. A bag contains 2 white and 1 red ball. One ball is drawn at random and then put back in the box after noting its colour. The process is repeated again. If 'X' denotes the number of red balls recorded in the two draws, describe 'X'.

Solution. Let the balls be w_1, w_2, r,

where $w \equiv$ white ball and $r \equiv$ red ball.

$\therefore$ Sample space S $= \{w_1w_1, w_1w_2, w_2w_1, w_2w_2, w_1r, rw_1, w_2r, rw_2, rr\}$.

For $\omega \in S$, X (ω) = Number of red balls.

$\therefore$ X $(\{w_1 w_1\})$ = X $(\{w_1 w_2\})$ = X $(\{w_2 w_1\})$

= X $(\{w_2 w_2\})$ = 0

X $(\{w_1 r\})$ = X $(\{rw_1\})$ = X $(\{w_2 r\})$

= X $(\{rw_2\})$ = 1 and X $(\{rr\})$ = 2.

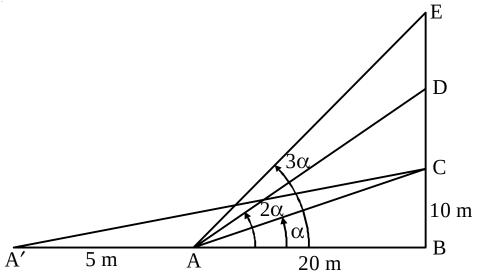
Hence, 'X' is a random variable, which takes values 0, 1, 2.

SECTION—E

(This section comprises of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36. The Government of India is planning to fix a hoarding board at the face of a building on the road of a busy market for awareness on COVID-19 protocol. Ram, Robert and Rahim are the three engineers who are working on this project. "A" is considered to be a person viewing the hoarding board 20 metres away from the building,

standing at the edge of a pathway nearby. Ram, Robert and Rahim suggested to the firm to place the hoarding board at three different locations namely, C, D and E. "C" is at the height of 10 metres from the ground level. For the viewer A, the angle of elevation of "D" is double the angle of elevation of "C". The angle of elevation of "E" is triple the angle of elevation of "C" for the same viewer.

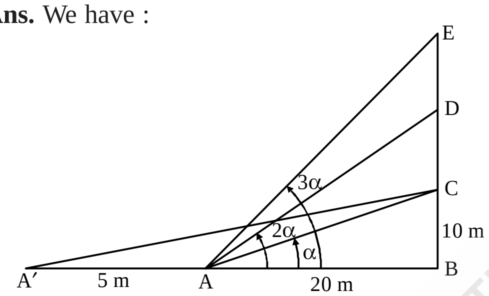


Look at the figure given and based on the above information answer the following:

(*a*) A′ is another viewer standing on the same line of observation across the road. If the width of the road is 5 meters, then find ∠CA′B.

(*b*) Find domain and range of $\tan^{-1} x$.

Ans. We have :



(*a*) $\tan \angle CA'B = \frac{10}{25} = \frac{2}{5}$

$\Rightarrow \quad \angle CA'B = \tan^{-1}\frac{2}{5}$.

(*b*) Domain of $\tan^{-1} x$ is **R** and

Range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Q. 37. Area under sample curves :

(*i*) Area of the region bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$.

(*ii*) Area of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = c$ and $y = d$ is $\int_c^d g(y)\,dy$.

Based on the above information, answer the following :

(*a*) Find the Area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

(*b*) Find the Area of the region bounded by the curve $y = 4x^2$, $x = 0$, $y = 1$ $y = 4$ in the first quadrant.

Ans. (*a*) Reqd. area $= 2\int_0^4 \sqrt{y}\,dy$

$$= 2\left[\frac{y^{3/2}}{3/2}\right]_0^4 = \frac{4}{3}\left[4^{3/2} - 0\right] = \frac{32}{3} \text{ sq. units}$$

(*b*) Reqd. area $= \int_1^4 \frac{\sqrt{y}}{2}\,dy = \frac{1}{2}\left[\frac{y^{3/2}}{3/2}\right]_1^4$

$$= \frac{1}{3}[8-1] = \frac{7}{3} \text{ sq. units.}$$

38. The equations of motion of a missile are $x = 3t$, $y = -4t$, $z = t$, where the time 't' is given in seconds, and the distance is measured in kilometres.

Based on the above answer the following :

(*a*) What is the path of the missile ?

(*b*) At what distance will the rocket be from the starting point (0, 0, 0) in 5 seconds ?

Ans. (*a*) St. line : $\frac{x}{3} = \frac{y}{-4} = \frac{z}{1} (= t)$

(*b*) At 5 secs, point is (15, – 20, 5).

Its distance from (0, 0, 0) $= \sqrt{(15)^2 + (-20)^2 + (5)^2}$

$$= \sqrt{225 + 400 + 25} = \sqrt{650} \text{ km.}$$

Holy Faith New Style Sample Paper–6

(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions : Same as Holy Faith New Style Sample Paper–1.

SECTION—A

(Multiple choice questions, each question carries 1 mark)

1. Let the relation R in the set A = $\{x \in \mathbf{Z} : 0 \le x \le 12\}$, given by :

R = $\{(a, b) : |a - b|$ is a multiple of 4$\}$.

Then {1}, the equivalence class containing 1, is :

(*a*) {1, 5, 9} (*b*) {0, 1, 2, 5}

(*c*) ϕ (*d*) A.

Ans. (*a*) {1, 5, 9}

2. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ is equal to :

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{3}$

(*c*) – 1 (*d*) 1.

Ans. (*d*) 1

3. If A = $\begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix}$ and A = A^T, where A^T is the transpose of the matrix A, then :

(*a*) $x = 0, y = 5$ (*b*) $x = y$

(*c*) $x + y = 5$ (*d*) $x = 5, y = 0$.

Ans. (*b*) $x = y$

4. If for a square matrix A, $A^2 - 3A + I = O$ and $A^{-1} = xA + yI$, then the value of $x + y$ is :

(*a*) – 2 (*b*) 2

(*c*) 3 (*d*) – 3.

Ans. (*b*) 2

5. The value of |A|, if A = $\begin{bmatrix} 0 & 2x-1 & \sqrt{x} \\ 1-2x & 0 & 2\sqrt{x} \\ -\sqrt{x} & -2\sqrt{x} & 0 \end{bmatrix}$,

where $x \in R^+$, is :x

(*a*) $(2x + 1)^2$ (*b*) 0

(*c*) $(2x + 1)^3$ (*d*) None of these.

Ans. (*b*) 0

6. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is continuous at :

(*a*) $x = 1$ (*b*) $x = 1.5$

(*c*) $x = -2$ (*d*) $x = 4$.

Ans. (*b*) $x = 1.5$

7. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, then $\frac{dy}{dx}$ is :

(*a*) $\frac{\cos\theta + \cos 2\theta}{\sin\theta - \sin 2\theta}$ (*b*) $\frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}$

(*c*) $\frac{\cos\theta - \cos 2\theta}{\sin\theta - \sin 2\theta}$ (*d*) $\frac{\cos 2\theta - \cos\theta}{\sin 2\theta + \sin\theta}$.

Ans. (*b*) $\frac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}$

8. In which of the interval the function $f(x) = 3x^2 - 4x$ is strictly increasing ?

(*a*) $(-\infty, 0)$ (*b*) (0, 2)

(*c*) $\left(\frac{2}{3}, \infty\right)$ (*d*) $(-\infty, \infty)$

Ans. (*a*) $(-\infty, 0)$

9. $\int \frac{dx}{x(x^2+1)}$ equals :

(*a*) $\log|x| - \frac{1}{2}\log(x^2+1) + c$

(*b*) $\frac{1}{2}\log|x| + \frac{1}{2}\log(x^2+1) + c$

(*c*) $-\log|x| + \frac{1}{2}\log(x^2+1) + c$

(*d*) $\log|x| + \log(x^2 + 1) + c$.

Ans. (*a*) $\log|x| - \frac{1}{2}\log(x^2+1) + c$

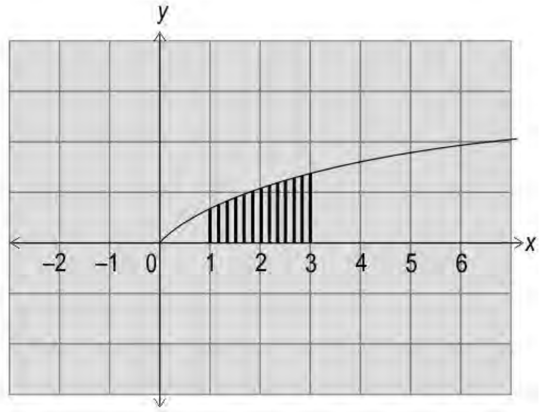
10. The value of $\int_0^{\pi/4} \sec 2x \, dx$ is:

(*a*) 0 (*b*) 1

(*c*) $\frac{1}{2}$ (*d*) $-\frac{1}{2}$.

Ans. (*b*) 1

11. Shown below is the curve defined by the equation $y = \log(x + 1)$ for $x \ge 0$.



Which of these is the area of the shaded region ?

(*a*) 6 log (2) – 2 (*b*) 6 log (2) – 6

(*c*) 6 log (2) (*d*) 5 log (2).

Ans. (*a*) 6 log (2) – 2

12. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is :

(*a*) $e^x + e^{-y} = c$ (*b*) $e^x + e^y = c$

(*c*) $e^{-x} + e^y = c$ (*d*) $e^{-x} + e^{-y} = c$.

Ans. (*a*) $e^x + e^{-y} = c$

13. The general solution of the differential equation: $xdy - (1 + x^2)\, dx = dx$ is:

(*a*) $y = 2x + \frac{x^3}{3} + c$ (*b*) $y = 2\log x + \frac{x^3}{3} + c$

(*c*) $y = \frac{x^2}{2} + c$ (*d*) $y = 2\log x + \frac{x^2}{2} + c$.

Ans. (*d*) $y = 2\log x + \frac{x^2}{2} + c$.

14. The value of '*p*' for which the vectors $2\hat{i} + p\hat{j} + \hat{k}$ and $-4\hat{i} - 6\hat{j} + 26\hat{k}$ are perpendicular to each other is :

(*a*) 3 (*b*) –3

(*c*) $\frac{-17}{3}$ (*d*) $\frac{17}{3}$.

Ans. (*a*) 3

15. The area of a triangle formed by vertices O, A and B, where $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\overrightarrow{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$ is :

(*a*) $3\sqrt{5}$ sq. units (*b*) $5\sqrt{5}$ sq. units

(*c*) $6\sqrt{5}$ sq. units (*d*) 4 sq. units.

Ans. (*a*) $3\sqrt{5}$ sq. units

16. The angle between the lines :

$2x = 3y = -z$ and $6x = -y = -4z$ is :

(*a*) 0° (*b*) 30°

(*c*) 45° (*d*) 90°.

Ans. (*d*) 90°

17. The feasible region of a linear programming problem is bounded. The corresponding objective function is : $Z = 6x - 7y$.

The objective function attains .... is the feasible region :

(*a*) only minimum

(*b*) only maximum

(*c*) both maximum and minimum

(*d*) either maximum or minimum but not both.

Ans. (*c*) both maximum and minimum

18. M and N are two events such that P (M ∩ N) = 0. Which of the following is equal to P (M/M ∪ N) ?

(*a*) $\frac{P(M)}{P(N)}$ (*b*) $\frac{P(M \cap N)}{P(M \cup N)}$

(*c*) $\frac{P(M)}{P(M) + P(N)}$ (*d*) $\frac{P(N)}{P(N) \times P(N)}$.

Ans. (*c*) $\frac{P(M)}{P(M) + P(N)}$

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.

(*b*) Both 'A' and 'R' are true but 'R' is not the correct explanation of 'A'.

(*c*) 'A' is true but 'R' is false.

(*d*) 'A' is false but 'R' is true.

Q. 19. Assertion (A) : The relation :

$f : \{1, 2, 3, 4\} \to \{x, y, z, p\}$ defined by :

$f : \{(1, x), (2, y), (3, z)\}$ is a bijective function.

Reason (R) : The function $f : \{1, 2, 3\} \to \{x, y, z\}$ such that $f\{(1, x), (2, y), (3, z)\}$ is one-one.

Ans. (*d*), **Assertion** (A) is false

[∵ *The element '4' has no image*]

Thus '*f*' is not a function.

Reason (R) is true because the function :

$f : \{1, 2, 3, 4\} \to \{x, y, z, p\}$ is one-one.

[∵ *For each* $x \in \{1, 2, 3\}$ *there is different image on* $\{x, y, z, p\}$ *under* f]

Q. 20. Assertion (A): Maximum value of Z = $3x + 4y$ subject to the constraints.

$x + y \le 4$, $x \ge 0$, $y \ge 0$ is 16.

Reason (R): Corner Point Method is not applicable.

Ans. (*c*);

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Show that :

$$\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}.$$

Solution. Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$.

$\therefore$ $\sin x = \frac{3}{5}$ and $\sin y = \frac{8}{17}$

so that $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

and $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$.

Now $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$$= \left(\frac{4}{5}\right)\left(\frac{15}{17}\right) + \left(\frac{3}{5}\right)\left(\frac{8}{17}\right)$$

$$= \frac{60+24}{85} = \frac{84}{85}$$

$\Rightarrow$ $x - y = \cos^{-1}\left(\frac{84}{85}\right)$.

Hence, $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$.

Or

Prove that :

$$\tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}.$$

Solution. Put $\cos^{-1}\frac{a}{b} = \theta$ so that $\cos\theta = \frac{a}{b}$.

$$\text{LHS} = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$= \frac{\left(1+\tan\frac{\theta}{2}\right)^2 + \left(1-\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{2(1+\tan^2\theta)}{1-\tan^2\frac{\theta}{2}} = \frac{2}{\cos 2.\frac{\theta}{2}} = \frac{2}{\cos\theta}$$

$$= \frac{2}{a/b} = \frac{2b}{a} = \textbf{RHS.}$$

Q. 22. Find the rate of change of the area of a circle with respect to its radius r when $r = 5$ cm.

Solution. Here 'r' is the radius of the circle.

Thus A, area of the circle = πr^2.

$\therefore$ $\frac{dA}{dr} = 2\pi r$.

Hence, $\left.\frac{dA}{dr}\right]_{r=5} = 2\pi(5) = 10\pi$ cm^2/s.

Q. 23. $f(x) = (2x-1)^2 + 3$

Solution. (*i*) We have : $f(x) = (2x-1)^2 + 3$.

Here D_f = **R.**

Now $f(x) \ge 3$. [$\because$ $(2x-1)^2 \ge 0$ *for all* $x \in$ **R**]

Hence, the minimum value = 3.

However, maximum value does not exist.

[$\because$ *f* (*x*) *can be made as large as we please*]

Or

Calculate the absolute maximum and absolute minimum value of the function f (x) = $\frac{x+1}{\sqrt{x^2+1}}$, $0 \le x \le 2$.

Solution. We have : $f(x) = \frac{x+1}{\sqrt{x^2+1}}$.

$\therefore$ $f'(x)$

$$= \frac{\sqrt{x^2+1}\,(1+0) - (x+1)\frac{1}{2\sqrt{x^2+1}}(2x+0)}{(x^2+1)}$$

$$= \frac{\sqrt{x^2+1} - \frac{x^2+x}{\sqrt{x^2+1}}}{x^2+1} = \frac{x^2+1-x^2-x}{(x^2+1)^{3/2}} = \frac{1-x}{(x^2+1)^{3/2}}.$$

Now $f'(x) = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1 \in [0, 2]$.

Now $f(0) = \frac{0+1}{\sqrt{0+1}} = \frac{1}{1} = 1$,

$f(1) = \frac{1+1}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

and $f(2) = \frac{2+1}{\sqrt{4+1}} = \frac{3}{\sqrt{5}}$.

Hence, absolute maximum value is $\frac{3}{\sqrt{5}}$ and absolute min. value is 1.

Q. 24. By using the properties of definite integral, evaluate the integral of $\int_2^8 |x-5|\,dx$.

Solution : Let $I = \int_2^8 |x-5|\,dx$

$$= \int_2^5 |x-5|\,dx + \int_5^8 |x-5|\,dx$$

$$= \int_2^5 -(x-5)\,dx + \int_5^8 (x-5)\,dx$$

$$\begin{bmatrix} \text{When } x<5 \Rightarrow x-5<0 \\ \Rightarrow |x-5| = -(x-5) \\ \text{When } x \geq 5 \Rightarrow x-5 \geq 0 \\ \Rightarrow |x-5| = x-5 \end{bmatrix}$$

$$\therefore I = -\left[\frac{x^2}{2}-5x\right]_2^5 + \left[\frac{x^2}{2}-5x\right]_5^8$$

$$= -\left[\left(\frac{25}{2}-25\right)-\left(\frac{4}{2}-10\right)\right]+\left[\left(\frac{64}{2}-40\right)-\left(\frac{25}{2}-25\right)\right]$$

$$= -\left[-\frac{25}{2}+8\right]+\left[-8+\frac{25}{2}\right]$$

$$= \frac{25}{2}-8-8+\frac{25}{2} = 25 - 16 = 9.$$

Q. 25. If $f(x) = \sqrt{\frac{\sec x - 1}{\sec x + 1}}$, find $f'\left(\frac{\pi}{3}\right)$.

Solution. We have : $f(x) = \sqrt{\frac{\sec x - 1}{\sec x + 1}}$

$$= \sqrt{\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}} = \sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$= \sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}} = \sqrt{\tan^2\frac{x}{2}} = \tan\frac{x}{2}.$$

$$\therefore f'(x) = \sec^2\frac{x}{2}.\frac{1}{2} = \frac{1}{2}\sec^2\frac{x}{2}.$$

Hence, $f'\left(\frac{\pi}{3}\right) = \frac{1}{2}\sec^2\frac{\pi}{6} = \frac{1}{2}(4) = 2$.

SECTION—C

(This section comprises of short answer type questions (SAQ) of 3 marks each)

Q. 26. Find : $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$.

Solution. Let $I = \int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$.

Put sin x = t so that $\cos x\,dx = dt$.

$$\therefore \quad I = \int \frac{2dt}{(1-t)(1+t^2)} \quad ...(1)$$

Let $\frac{2}{(1-t)(1+t^2)} \equiv \frac{A}{1-t}+\frac{Bt+C}{1+t^2}$ ...(2)

Multiplying by $(1-t)(1+t^2)$, we get :

$$2 \equiv A(1+t^2)+(Bt+C)(1-t).$$

Putting $t = 1$, $2 = A(2) + 0$

$\Rightarrow$ A = 1.

Comparing coeffs. of t^2, $0 = A - B$

$\Rightarrow$ B = A = 1.

Comparing coeffs. of t, $0 = B - C$

$\Rightarrow$ C = B = 1.

$\therefore$ From (2), $\frac{2}{(1-t)(1+t^2)} = \frac{1}{1-t}+\frac{t+1}{t^2+1}$.

$\therefore$ From (1), $I = \int \frac{1}{1-t}dt + \int \frac{t+1}{t^2+1}dt$

$$= -\log|1-t| + \frac{1}{2}\int \frac{2t}{t^2+1}dt + \int \frac{1}{t^2+1}dt$$

$$= -\log|1-t| + \frac{1}{2}\log|t^2+1| + \tan^{-1}t + c$$

$$= -\log|1-t| + \frac{1}{2}\log(t^2+1) + \tan^{-1}t + c$$

$[\because t^2 \geq 0 \Rightarrow t^2+1>0 \Rightarrow |t^2+1| = t^2+1]$

$$= -\log|1-\sin x| + \frac{1}{2}\log(\sin^2 x + 1) + \tan^{-1}(\sin x) + c$$

$$= -\log(1-\sin x) + \frac{1}{2}\log(1+\sin^2 x) + \tan^{-1}(\sin x) + c.$$

[$\because$ $1 - \sin x \geq 0 \Rightarrow |1 - \sin x| = 1 - \sin x$]

Q. 27. A person plays a game of tossing a coin thrice. For each head he is given ₹ 2 by the organiser of the game and for each tail he has to give ₹ 1·50 to the organiser. Let 'X' denote the amount gained or lost by the person. Show that 'X' is a random variable and exhibit it as a function on the sample space of the experiment.

Solution. Since 'X' is a number whose values are defined by the outcomes of the random experiment,

$\therefore$ 'X' is a random variable.

Now sample space is given by :

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT},

where H ≡ Head and T ≡ Tail.

Thus X (HHH) = 2 × 3 = ₹ 6

X (HHT) = X (HTH) = X (THH)
= 2 × 2 – 1 × 1·5 = ₹ 2·50

X (HTT) = X (THT) = X (TTH)
= 1 × 2 – 2 × 1·5 = – ₹ 1

and X (TTT) = – (3 × 1·5) = – ₹ 4·50.

Thus for each element of S, X takes a unique value.

$\therefore$ 'X' is a function on the sample space S having range = {6, 2·50, – 1, – 4·50}.

Q. 28. Find : $\int \frac{\tan^2 x \sec^2 x}{1 - \tan^6 x} dx$.

Solution. $I = \int \frac{\tan^2 x \sec^2 x}{1-(\tan^3 x)^2} dx.$

Put $\tan^3 x = t$ so that $3\tan^2 x . \sec^2 x \; dx = dt$

i.e. $\tan^2 x \sec^2 x \, dx = \frac{dt}{3}.$

$$\therefore \quad I = \int \frac{\frac{1}{3}dt}{1-t^2}$$

$$= \frac{1}{3}\int \frac{dt}{1-t^2} \quad \Big| \text{"Form : } \int \frac{1}{a^2-x^2}dx\text{"}$$

$$= \frac{1}{3} \cdot \frac{1}{2(1)} \log\left|\frac{1+t}{1-t}\right| + c$$

$$= \frac{1}{6}\log\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right| + c.$$

Or

Prove $\int_1^4 |x-1| + |x-2| + |x-3|] \, dx$.

Solution : $I = \int_1^4 (|x-1| + |x-2| + |x-3|)\, dx$

$$= \int_1^4 |x-1|\,dx + \int_1^4 |x-2|\,dx + \int_1^4 |x-3|\,dx \quad ...(1)$$

Now $\int_1^4 |x-1|\,dx$ [$\because$ $|x-1| = x-1$ *when* $x \geq 1$ *i.e. when* $x - 1 \geq 0$]

$$\therefore \int_1^4 |x-1|\,dx = \left[\frac{x^2}{2} - x\right]_1^4$$

$$= \left(\frac{16}{2} - 4\right) - \left(\frac{1}{2} - 1\right)$$

$$= 4 + \frac{1}{2} = \frac{9}{2}.$$

And $\int_1^4 |x-2|\,dx$

$$= \int_1^2 |x-2|\,dx + \int_2^4 |x-2|\,dx$$

$$= \int_1^2 -(x-2)\,dx + \int_2^4 (x-2)\,dx$$

[$\because$ $|x-2| = -(x-2)$ *when* $x \leq 2$
i.e. when $x - 2 \leq 0$
and $|x-2| = (x-2)$ *when* $x \geq 2$
i.e. when $x - 2 \geq 0$]

$$= \left[\frac{x^2}{2} - 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$= -\left[\left(\frac{4}{2}-4\right)-\left(\frac{1}{2}-2\right)\right]+\left[\left(\frac{16}{2}-8\right)-\left(\frac{4}{2}-4\right)\right]$$

$$= -\left[-2+\frac{3}{2}\right]+[0+2] = \frac{1}{2}+2 = \frac{5}{2}.$$

Finally, $\int_1^4 |x-3|\,dx = \int_1^3 |x-3|\,dx + \int_3^4 |x-3|\,dx$

$$= \int_1^3 -(x-3)\,dx + \int_3^4 (x-3)\,dx$$

[$\because$ *$|x-3| = -(x-3)$ when $x \le 3$ i.e. when $x-3 \le 0$ and $|x-3| = (x-3)$ i.e. when $x \ge 3$ i.e. when $x-3 \ge 0$*]

$$= -\left[\frac{x^2}{2}-3x\right]_1^3 + \left[\frac{x^2}{2}-3x\right]_3^4$$

$$= -\left[\left(\frac{9}{2}-9\right)-\left(\frac{1}{2}-3\right)\right]+\left[\left(\frac{16}{2}-12\right)-\left(\frac{9}{2}-9\right)\right]$$

$$= -\left[\frac{-9}{2}+\frac{5}{2}\right]+\left[-4+\frac{9}{2}\right] = 2+\frac{1}{2} = \frac{5}{2}.$$

$\therefore$ From (1), $\int_1^4 f(x)\,dx = \frac{9}{2}+\frac{5}{2}+\frac{5}{2} = \frac{19}{2}.$

Prove the following Exercises 32 to 37.

Q. 29. Find : $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$.

Solution : We have : $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$.

Integrating, $y = \int \frac{1-\cos x}{1+\cos x}\,dx + c$

$$= \int \frac{2\sin^2 x/2}{2\cos^2 x/2}\,dx + c$$

$$= \int \tan^2 \frac{x}{2}\,dx + c$$

$$= \int \left(\sec^2 \frac{x}{2} - 1\right) dx + c$$

$$= \frac{\tan x/2}{1/2} - x + c$$

$\Rightarrow$ $y = 2\tan\frac{x}{2} - x + c,$

which is the reqd. general solution.

Or

Find the general solution of $(x+3y^2)\frac{dy}{dx} = y\ (y > 0)$.

Solution : The given equation is $(x+3y^2)\frac{dy}{dx} = y$

$\Rightarrow$ $x + 3y^2 = y\frac{dx}{dy}$

$\Rightarrow$ $\frac{dx}{dy} - \frac{x}{y} = 3y$ ...(1) | ***Linear in x***

Here 'P' = $-\frac{1}{y}$ and 'Q' = $3y$.

$\therefore$ I.F. $= e^{\int P\,dy} = e^{-\int \frac{1}{y}dy} = e^{-\log y}$

$= e^{\log y^{-1}} = y^{-1} = \frac{1}{y}.$

Multiplying (1) by $\frac{1}{y}$, we get :

$$\frac{1}{y}\cdot\frac{dx}{dy} - \frac{x}{y^2} = 3$$

$\Rightarrow$ $\frac{d}{dy}\left(x\frac{1}{y}\right) = 3.$

Integrating, $x\cdot\frac{1}{y} = 3\int 1\,.\,dy + c$

$\Rightarrow$ $\frac{x}{y} = 3y + c$

$\Rightarrow$ $x = 3y^2 + cy,$

which is the reqd. solution.

Q. 30. A small firm manufactures neclaces and bracelets. The total number of neclaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a neclace. The maximum number of hours available per day is 16. If the profit on a neclace is ₹ 100 and that on a bracelet is ₹ 300. Formulate an LPP for finding how many of each should be produced daily to maximize the profit ?

It is being given that at least one of each must be produced.

Solution. Let 'x' neclaces and 'y' bracelets be manufactured per day.

Then LPP problem is :

Maximize : Z = 100 x + 300 y

Subject to the constraints :

$x + y \le 24$,

(1) $(x) + \frac{1}{2}y \le 16$, *i.e.* $2x + y \le 32$

and $x \ge 1$ and $y \ge 1$ *i.e.* $x - 1 \ge 0$ and $y - 1 \ge 0$.

Or

If A = $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -26 \end{bmatrix}$**, find** A^{-1}**.**

Using A^{-1}**, solve the system of equations :**

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 2$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z} = 5$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z} = -4.$$

Solution. The given system of equations is :

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 2 \quad ...(1)$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z} = 5 \quad ...(2)$$

and $$\frac{6}{x}+\frac{9}{y}-\frac{20}{z} = -4 \quad ...(3)$$

These equations can be written as AX = B ...(4),

where $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, $X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$.

Now $|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$

= 2(120 – 45) – 3 (– 80 – 30) + 10 (36 + 36)

= 150 + 330 + 720 = 1200 $\neq$ 0.

$\therefore$ A is non-singular and as such A^{-1} exists.

Now $A_{11} = \begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} = 120 - 45 = 75;$

$A_{12} = -\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} = -(-80 - 30) = 110;$

$A_{13} = \begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix} = 36+36 = 72.$

$A_{21} = -\begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix} = -(-60-90) = 150;$

$A_{22} = \begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix} = -40-60 = -100;$

$A_{23} = -\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = -(18-18) = 0.$

$A_{31} = \begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix} = 15+60 = 75;$

$A_{32} = -\begin{vmatrix} 2 & 10 \\ 4 & 5 \end{vmatrix} = -(10-40) = 30;$

$A_{33} = \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} = -12-12 = -24.$

$$\therefore \text{adj } A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}' = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

From (4), $A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow \quad IX = A^{-1}B \Rightarrow X = A^{-1}B.$

$$\therefore \quad X = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 150+750-300 \\ 220-500-120 \\ 144+0+96 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ -400 \\ 240 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} 1/x \\ 1/y \\ -1/z \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/3 \\ 1/5 \end{bmatrix}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{2}, \; \frac{1}{y} = -\frac{1}{3} \text{ and } \frac{1}{z} = \frac{1}{5}.$$

Hence, $x = 2, y = -3$ and $z = 5$.

Q. 31. When an object is thrown vertically upword it is under the effect of gravity and air resistance. For small objects, the focus due to air resistance is numerically equal to some constant k times v, where v is the velocity of the object (in m/s) in time t (s). The situation can be modelled as the differential equation shown below.

$$\frac{dv}{dt} = -F_R - mg,$$

which m is the mass of the object in kg. $\frac{dx}{dt}$ **is the acceleration of the object in** m/s^2**.**

F_R **is the force due to air resistance, g is the accerlation due to gravity (10** m/s^2**).**

A tennis ball of mass 0.050 kg. is but upwords with a velocity of 10 m/s. An air resistance merically equal to 0.400 acts on the ball.

(i) Model the above situation, using differated equation

(ii) Write an expression for the velocity of the ball in tenis of the time. Show your words.

Solution. (*i*) Models for the situation as as below :

$$0.05\frac{dv}{dt} = -0.4v - 0.5$$

$$\Rightarrow \quad \frac{dx}{dt} + 8v = 0 \qquad \text{....(1) | \textit{Linear Equation}}$$

(*i*) Here P = 8 and Q = 10 [Composing and $\frac{dy}{dt}$, Py = Q]

$$\therefore \quad \text{I.F.} = e^{\int P\,dt}$$

$$= e^{\int 8\,dt} = e^{8t}$$

Multiplying (1) by e^{8t}, we get :

$$e^{8t}\frac{dv}{dt} + 8ve^{8t} = -10e^{8t}$$

$$\Rightarrow \quad \frac{d}{dt}(ve^{8t}) = -10e^{8t}$$

Integration, $$\text{V}e^{8t} = -10\int e^{8t} + c$$

$$\Rightarrow \quad \text{V}e^{8t} = \frac{-10}{8}e^{8t} + c$$

$$\Rightarrow \quad e = -1.25 + ce^{-2t} \qquad \text{....(2)}$$

When V (0) = 10, them

$$10 = -1.25 + c \Rightarrow c = 11.25$$

Putting in (2), V = – 1.25 + 11.25 e^{-8t}, which is the reqd. expression.

SECTION—D

(This section comprises of Long Answer Type questions (LAQ) of 5 marks each)

Q. 32. Find the area enclosed by the ellipse : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b.}$

Solution. The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(1)

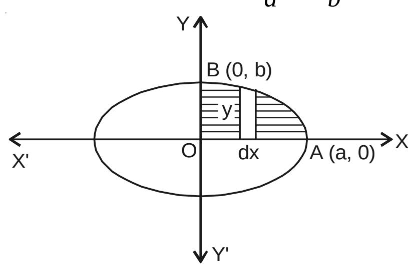


Fig.

Area of the ellipse = 4 (area of the region OABO, bounded by the curve, x-axis and the ordinates $x = 0$, $x = a$)

[∵ *Ellipse is symmetrical about both the axes*]

$$= 4\int_0^a y\,dx \qquad \text{[\textit{Taking vertical strips}]}$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

[∵ (*1*) ⇒ $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$. *But region OABO lies in 1st quadrant,* ∴ *y is +ve*]

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[\left\{\frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1)\right\} - \{0 + 0\}\right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right] = \pi ab \text{ sq. units.}$$

Q. 33. Let A = {1, 2, 3,........., 9} and *R* be the relation in A × A defined by (a, b) *R* (c, d) if :

a + d = b + c for (a, b), (c, d) in A × A.

Prove that *R* is an equivalence relation.

Also, obtain the equivalence class {(2, 5)}.

Solution. (*i*) We have : (a, b) R $(c, d) \Rightarrow a + d = b + c$ on the set A = {1, 2, 3, ..., 9}.

(I) (a, b) R $(a, b) \Rightarrow a + b = b + a$, which is true.

[∵ $a + b = b + a\ \forall\, a, b \in A$]

Thus *R is reflexive.*

(II) (a, b) R $(c, d) \Rightarrow a + d = b + c$

(c, d) R $(a, b) \Rightarrow c + b = d + a.$

But $c + b = b + c$ and $d + a = a + d\ \forall\ a, b, c, d \in$ A.

∴ (a, b) R (c, d) = (c, d) R (a, b).

Thus *R is symmetric.*

(III) (a, b) R $(c, d) \Rightarrow a + d = b + c\ \ \forall\ a, b, c, d \in$ A ...(1)

(c, d) R $(e, f) \Rightarrow c + f = d + e\ \ \forall\ c, d, e, f \in$ A ...(2)

Adding (1) and (2),

$$(a + d) + (c + f) = (b + c) + (d + e)$$

$$\Rightarrow \quad a + f = b + e \Rightarrow (a, b)\text{ R }(e, f).$$

Thus *R is transitive.*

Hence, the relation R is an equivalence relation.

(*ii*) {(2, 5)} = {(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)}.

[∵ *2 + 4 = 5 + 1; etc.*]

Or

Show that the function $f : \mathbf{R} \to \{x \in \mathbf{R} : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}, x \in \mathbf{R}$ is one-one and onto function.

Solution. We have : $f(x) = \begin{cases} \dfrac{x}{1+x}, & \text{if } x \geq 0 \\ \dfrac{x}{1-x}, & \text{if } x < 0. \end{cases}$

Three cases arise :

Case I: When $x \geq 0$.

Here, $f(x) = \dfrac{x}{1+x}$.

Injectivity:

Let $x, y \in \mathbf{R}^+ \cup \{0\}$ such that $f(x) = f(y)$

$\Rightarrow \quad \dfrac{x}{1+x} = \dfrac{y}{1+y}$

$\Rightarrow \quad x + xy = y + xy$

$\Rightarrow \quad x = y.$

Thus, f is ***injective*** function.

Surjectivity:

When $x \geq 0$, then $f(x) = \dfrac{x}{1+x} \geq 0$

$\Rightarrow \quad f(x) = 1 - \dfrac{1}{1+x} < 1 \quad [\because x > 0]$

Let $y \in [0, 1)$. Then

$x = \dfrac{y}{1-y}$ such that $f(x) = \dfrac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = y.$

Thus, f is onto function an $[0, \infty)$ to $[0,1)$.

Case II : When $x < 0$.

Here, $f(x) = \dfrac{y}{1-y}$

Injectivity:

Let $x, y \in R^-$ i.e., $x, y < 0$ such that $f(x) = f(y)$

$\Rightarrow \quad \dfrac{x}{1+x} = \dfrac{y}{1-y}$

$\Rightarrow \quad x - xy = y - xy$

$\Rightarrow \quad x = y$

$\Rightarrow f$ is injevtive function.

Subjectivity :

When $x < 0$, then $f(x) = \dfrac{x}{1-x} < 0$

Also, $f(x) = \dfrac{x}{1-x} = -1 + \dfrac{x}{1-x} > -1$

$\Rightarrow \quad -1\, f(x) < 0.$

Let $y \in (-1, 0)$ be an arbitrary real number.

Then what exists $x = \dfrac{y}{1+y} < 0$ such that,

$f(x) = f\left(\dfrac{y}{1+y}\right) = \dfrac{\frac{y}{1+y}}{1-\frac{y}{1+y}} = y.$

Thus, for $y \in (-1, 0)$ there exists $x = \dfrac{y}{1+y} < 0$ such that $f(x) = y$.

Hence, f is onto function on $(-\infty, 0)$ to $(-1, 0)$

Case III : Let $x > 0$ and $y < 0$ such that

$f(x) = f(y)$

$\Rightarrow \quad \dfrac{x}{1-x} = \dfrac{y}{1-y}$

$\Rightarrow \quad x - xy = y + xy$

$\Rightarrow \quad x - y = 2xy$

Here, **LHS** > 0 but **RHS** < 0.

Thus, there is contradictor.

Here, $f(x) \neq f(y)$ when $x \neq y$.

Hence, f is one-one and onto function.

Q. 34. Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (–5, 7).

Solution : Let A (2, 1) and B (– 5, 7) be the initial and terminal points respectively.

$\therefore \quad \overrightarrow{AB} = \text{P.V. of B} - \text{P.V. of A}$

$= (-5\hat{i} + 7\hat{j}) - (2\hat{i} + \hat{j})$

$= -7\hat{i} + 6\hat{j}.$

Hence, – 7 and 6 are scalar components and $-7\hat{i}$ and $6\hat{j}$ are vector components of $\overrightarrow{AB}$.

Or

Find the shortest distance between the lines :

$$\vec{\mathbf{r}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

and $$\vec{\mathbf{r}} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \mu(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}).$$

Solution. Comparing given equations with :

$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$, we have :

$\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}, \ \vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$

and $\vec{a}_1 = 3\hat{i} + 2\hat{j} - 4\hat{k}, \ \vec{a}_2 = 5\hat{i} - 2\hat{j}.$

$\therefore \quad \vec{a}_2 - \vec{a}_1 = 2\hat{i} - 4\hat{j} + 4\hat{k}.$

Now, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 3 & 2 & 6 \end{vmatrix}$

$= \hat{i}(12-4) - \hat{j} + (6-6) + \hat{k}(2-6)$

$= 8\hat{i} - 4\hat{k}.$

Since $(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1}) = (8\hat{i} - 4\hat{k}).(2\hat{i} - 4\hat{j} + 4\hat{k})$

$= 16 - 16 = 0,$

$\therefore$ the lines are intersecting and the shortest distance between the lines is 0.

For the point of intersection :

$$3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$

$$= 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

$\Rightarrow \quad 3 + \lambda = 5 + 3\mu \qquad$ ...(1)

$2 + 2\lambda = -2 + 2\mu \qquad$ ...(2)

and $\ -4 + 2\lambda = 6\mu \qquad$ ...(3)

Solving (1) and (2), we get : $\lambda = -4$ and $\mu = -2$.

Putting in the equation of the line, we get :

$$\vec{r} = 5\hat{i} - 2\hat{j} + (-2)(3\hat{i} + 2\hat{j} + 6\hat{k})$$

$$= -\hat{i} - 6\hat{j} - 12\hat{k}.$$

Hence, the point of intersection is (– 1, – 6, – 12).

Q. 35. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A ?

Solution : Let the events be :

E_1 : Item is from machine A

E_2 : Item is from machine B

E_3 : Item is from machine C

and A : Items is defective.

$\therefore \quad P(E_1) = \dfrac{50}{100}, \ P(E_2) = \dfrac{30}{100}$ and

$P(E_3) = \dfrac{20}{100}.$

Also $P(A/E_1) = \dfrac{1}{100}, \ P(A/E_2) = \dfrac{5}{100}$

and $P(A/E_3) = \dfrac{7}{100}.$

By *Bayes' Theorem,*

$P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{50}{100} \times \frac{1}{100}}{\frac{50}{100} \times \frac{1}{100} + \frac{30}{100} \times \frac{5}{100} + \frac{20}{100} \times \frac{7}{100}}$$

$$= \frac{50}{50 + 150 + 140} = \frac{50}{340} = \frac{5}{34}.$$

SECTION—E

(This section comprises of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36. Raji visited the Exhibition along with her family. The Exhibition had a huge swing, which attracted many children. Raji found that the swing traced the path of a parabola as given by $y = x^2$.

Based on the above information, answer the following:

(*a*) Let $f : \{1, 2, 3, \ldots\} \to \{1, 4, 9, \ldots\}$ be defined by $f(x) = x^2$. Then R is neither surjective nor injective.

(*b*) The function $f : Z \to Z$ defined by $f(x) = x^2$ is neither injective nor surjective.

Ans. (*a*) Yes, R is neither surjective nor injective

(*b*) Yes, R is neither injective nor surjective.

Q. 37. Three schools DPS, CVC and KVS decided to organize a fair for collecting money for helping the flood victims. They sold handmade fans, mats and plates from recycled material at a cost of ₹ 25, ₹ 100 and ₹ 50 each respectively. The number of articles sold are given as :

School/Article	DPS	CVC	KVS
Handmade fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Based on the above information, answer the following :

(I) What is the total money (in ₹) collected by the school DPS ?

(II) What is the total amount of money (in ₹) collected by the schools CVC and KVS?

(III) What is the total amount of money (in ₹) collected by all three schools DPS, CVC and KVS ?

Or

How many articles (in total) are sold by three schools ?

Ans. (I) ₹ 5,250

(II) ₹ 13,250

(II) ₹ 18,500 **Or** 350.

Q. 38. A veterinary doctor was examining a sick cat brought by a pet lover. When it was brought to the hospital, it was already dead. The pet lover wanted to find its time of death. He took the temperature of the cat at 11:30 pm which was 94.6°F. He took the temperature again after one hour; the temperature was lower than the first observation. It was 93.4°F. The room in which the cat was put is always at 70°F. The normal temperature of the cat is taken as 98.6°F when it was alive. The doctor estimated the time of death using *Newton law of cooling which is governed by the differential equation* : $\frac{dT}{dt} \propto (T - 70)$, where 70°F is the room temperature and T is the temperature of the object at time *t*.

Substituting the two different observations of T and *t* made, in the solution of the differential equation $\frac{dT}{dt} = k(T - 70)$, where *k* is a constant of proportion, time of death is calculated.

(I) State the degree of the above given differential equation.

(II) Which method of solving a differential equation helped in calculation of the time of death ?

(III) If the temperature was measured 2 hours after 11.30 pm, will the time of death change ? (Yes/No)

Or

Find the solution of the differential equation $\frac{dT}{dt} = k(T - 70)$.

Ans. (I) Degree is 1.

(II) (*a*) Variable separable method

(III) No.

Or

(*a*) $\frac{dT}{T - 70} = k\,dt$

$\Rightarrow \int \frac{dT}{T - 70} = k \int dt$

$\Rightarrow \log |T - 70| = kt + c.$

Holy Faith New Style Sample Paper–7

(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions : Same as Holy Faith New Style Sample Paper–1.

SECTION—A

(Multiple choice questions, each question carries 1 mark)

1. The function $f : \mathbf{R} \rightarrow \mathbf{R}$ defined as $f(x) = x^3$ is :

(*a*) one-one but not onto
(*b*) not one-one but onto
(*c*) neither one-one nor onto
(*d*) one-one and onto.

Ans. (*d*) one-one and onto.

2. $\sin(\tan^{-1} x)$, where $|x| < 1$, is equal to :

(*a*) $\frac{x}{\sqrt{1-x^2}}$ (*b*) $\frac{1}{\sqrt{1-x^2}}$

(*c*) $\frac{1}{\sqrt{1+x^2}}$ (*d*) $\frac{x}{\sqrt{1+x^2}}$.

Ans. (*d*) $\frac{x}{\sqrt{1+x^2}}$.

3. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $(3I + 4A)(3I - 4A) = x^2 I$, then the value (*s*) of x is/are :

(*a*) $\pm\sqrt{7}$ (*b*) 0
(*c*) ± 5 (*d*) 25.

Ans. (*a*) $\pm\sqrt{7}$.

4. If $\left|\frac{A^{-1}}{2}\right| = \frac{1}{k|A|}$, where A is a 3×3 matrix, then the value of k is :

(*a*) $\frac{1}{8}$ (*b*) 8

(*c*) 2 (*d*) $\frac{1}{2}$.

Ans. (*a*) $\frac{1}{8}$.

5. Given that A is a square matrix of order 3 and $|A| = -2$, then $|\text{adj}(2A)|$ is equal to :

(*a*) -2^6 (*b*) 4
(*c*) -2^8 (*d*) 2^8.

Ans. (*a*) -2^6.

6. The value of 'k' for which :

$$f(x) = \begin{cases} 3x + 5, & x \geq 2 \\ kx^2, & x < 2 \end{cases}$$ is continuous function, is :

(*a*) $-\frac{11}{4}$ (*b*) $\frac{4}{11}$

(*c*) 11 (*d*) $\frac{11}{4}$.

Ans. (*c*) 11.

7. If $f(x) = \cos^{-1}\sqrt{x}$, $0 < x < 1$, which of the following is equal to $f'(x)$?

(*a*) $\frac{-1}{\sqrt{1-x}}$ (*b*) $\frac{1}{\sqrt{1-x}}$

(*c*) $\frac{1}{2\sqrt{x(1-x)}}$ (*d*) $\frac{-1}{2\sqrt{x(1-x)}}$.

Ans. (*a*) $\frac{-1}{\sqrt{1-x}}$.

8. The function $y = x^2 e^{-x}$ is decreasing in the interval :

(*a*) (0, 2) (*b*) $(2, \infty)$
(*c*) $(-\infty, 0)$ (*d*) $(-\infty, 0) \cup (2, \infty)$.

Ans. (*b*) $(2, \infty)$.

9. $\int \frac{x\,dx}{(x-1)(x-2)}$ equals :

(*a*) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + c$ (*b*) $\log\left|\frac{(x-2)^2}{x-1}\right| + c$

(*c*) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + c$ (*d*) $\log|(x-1)(x-2)| + c$.

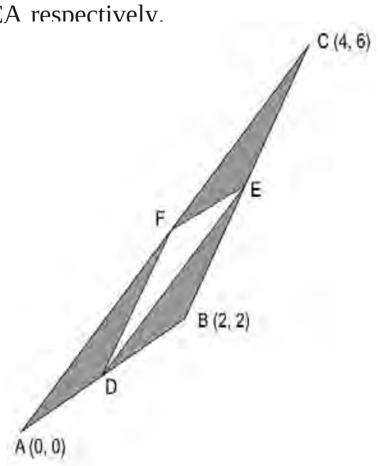
Ans. (*d*) $\log|(x-1)(x-2)| + c$.

10. $\int_{-1}^{1} \frac{|x+2|}{x-2}dx$, $x \neq 2$ is equal to :

(*a*) 1 (*b*) –1
(*c*) 2 (*d*) –2.

Ans. (*a*) 1.

11. The points D, E and F are the mid-points of AB, BC and CA respectively.



What is the area of the shaded region ?

(*a*) 2 sq. units (*b*) $\frac{3}{2}$ sq. units

(*c*) $\frac{1}{2}$ sq. units (*d*) $(2\sqrt{26}-1)$ sq. units.

Ans. (*d*) $(2\sqrt{26}-1)$ sq. units.

12. The Integrating factor of the differential equation :

$(1-y^2)\frac{dx}{dy} + yx = ay \quad (-1 < y < 1)$ is :

(*a*) $\frac{1}{y^2-1}$ (*b*) $\frac{1}{\sqrt{y^2-1}}$

(*c*) $\frac{1}{1-y^2}$ (*d*) $\frac{1}{\sqrt{1-y^2}}$.

Ans. (*c*) $\frac{1}{1-y^2}$.

13. The solution of the differential equation :

$x\frac{dy}{dx} + 2y = x^2$ is :

(*a*) $y = \frac{x^2+c}{4x^2}$ (*b*) $y = \frac{x^2}{4} + c$

(*c*) $y = \frac{x^4+c}{x^2}$ (*d*) $y = \frac{x^4+c}{4x^2}$.

Ans. (*d*) $y = \frac{x^4+c}{4x^2}$.

14. If $\vec{a}.\hat{i} = \vec{a}.(\hat{i}+\hat{j}) = \vec{a}.(\hat{i}+\hat{j}+\hat{k}) = 1$, then $\vec{a}$ is :

(*a*) $\hat{k}$ (*b*) $\hat{i}$

(*c*) $\hat{j}$ (*d*) $\hat{i}+\hat{j}+\hat{k}$.

Ans. (*b*) $\hat{i}$.

15. Direction-cosines of a line perpendicular to both *x*-axis and *z*-axis are :

(*a*) < 1, 0, 1 > (*b*) < 1, 1, 1 >
(*c*) < 0, 0, 1 > (*d*) < 0, 1, 0 >

Ans. (*d*) < 0, 1, 0 >.

16. A Linear Programming Problem is as follows :

Minimise $Z = 2x + y$
subject to the constraints $x \geq 3, x \leq 9, y \geq 0$
$x - y \geq 0, x + y \leq 14$.

The feasible region has :
(*a*) 5 corner points including (0, 0) and (9, 5)
(*b*) 5 corner points including (7, 7) and (3, 3)
(*c*) 5 corner points including (14, 0) and (9, 0)
(*d*) 5 corner points including (3, 6) and (9, 5).

Ans. (*b*) 5 corner points including (7, 7) and (3, 3).

17. The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is :

(*a*) 1 (*b*) 2

(*c*) 5 (*d*) $\frac{8}{3}$.

Ans. (*b*) 2.

18. Let A and B be two events such that P(A) = 0.6, P(B) = 0.2 and P(A/B) = 0.5. Then P(A′/B′) equals :

(*a*) $\frac{1}{10}$ (*b*) $\frac{3}{10}$

(*c*) $\frac{3}{8}$ (*d*) $\frac{6}{7}$.

Ans. (*c*) $\frac{3}{8}$.

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) **Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.**
(*b*) **Both 'A' and 'R' are true but 'R' is not the correct explanation of 'A'.**
(*c*) **'A' is true but 'R' is false.**
(*d*) **'A' is false but 'R' is true.**

Q. 19. If $\int f(x)\,dx = F(x) - c$, then :

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Statement–(A) : $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$.

Statement–(R) : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

and $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$,

if $f(2a-x) = f(x)$.

Ans. (*a*) Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.

Q. 20. Assertion (A) : If $|\vec{a}| = 2, |\vec{b}| = 3, |2\vec{a} - \vec{b}| = 5$, then $|2\vec{a} + \vec{b}| = 5$.

Reason (R) : $|\vec{a} - \vec{b}| = |\vec{a} + \vec{b}|$.

Ans. (*c*) 'A' is true but 'R' is false.

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Reduce : $\cot^{-1}\left\{\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right\}$**, where** $\dfrac{\pi}{2} < x < \pi$ **into simplest form.**

Ans. $\cot^{-1}\left\{\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right\}$

$$= \cot^{-1}\left\{\frac{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} + \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}}{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} - \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}}\right\}$$

$$= \cot^{-1}\left\{\frac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)}\right\}$$

$$= \cot^{-1}\left\{\frac{2\sin\frac{x}{2}}{2\cos\frac{x}{2}}\right\} = \cot^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \cot^{-1}\left[\cot\left(\frac{\pi}{2} - \frac{x}{2}\right)\right] = \frac{\pi}{2} - \frac{x}{2}.$$

Or

Write $\cot^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right)$**,** $x > 1$ **in the simplest from.**

Ans. Let $x = \sec\theta$. Then $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$.

$$\therefore \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \cot^{-1}(\cot\theta) = \theta = \sec^{-1} x.$$

Q. 22. The total revenue in (₹) received from the sale of x units of a product is given by R(x) = $3x^2 + 36x + 5$. Find the marginal revenue, when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.

Ans. $\text{MR} = \dfrac{d\text{R}}{dx} = \dfrac{d}{dx}(3x^2 + 36x + 5)$

$= 6x + 36.$

When $x = 5$, $\text{MR} = 6(5) + 36 = 30 + 36 = 66.$

Hence, the required marginal revenue is ₹ 66.

Q. 23. Find the maximum and minimum values of : $fx = 16x^2 - 16x + 28$ without using derivative.

Ans. We have : $f(x) = 16x^2 - 16x + 28.$

Here $D_f = \mathbf{R}.$

Now $f(x) = 16\left(x^2 - x + \frac{1}{4}\right) + 24$

$= 16\left(x - \frac{1}{2}\right)^2 + 24$

$\Rightarrow f(x) \geq 24.$

$\left[\because 16\left(x - \frac{1}{2}\right)^2 \geq 0 \text{ for all } x \in \mathbf{R}\right]$

Hence, the minimum value is 24.

However, maximum value does not exist.

[$\because$ *$f(x)$ can be made as large as we please*]

Or

Prove that $h(x) = x^3 + x^2 + x + 1$ do not have maxima or minima.

Ans. We have : $f(x) = x^3 + x^2 + x + 1.$

$\therefore f'(x) = 3x^2 + 2x + 1.$

Now $f'(x) = 0$

$\Rightarrow 3x^2 + 2x + 1 = 0$

$\Rightarrow x = \dfrac{-2 \pm \sqrt{4-12}}{6} = \dfrac{-2 \pm \sqrt{-8}}{6},$

which are non-real.

Hence, $f(x)$ does not have maximum and minimum values.

Q. 24. Using the properties of definite integral evaluate the integral $\displaystyle\int_0^1 x(1-x)^n dx$

Ans. $\displaystyle I = \int_0^1 x(1-x)^n \, dx$ ...(1)

$$= \int_0^1 (1-x)[1-(1-x)]^n \, dx$$

$$\left[\because \int_b^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)\,x^n \, dx = \int_0^1 (x^n - x^{n+1})\,dx$$

$$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$$

$$= \left[\frac{1}{n+1} - \frac{1}{n+2}\right] - (0-0)$$

$$= \frac{1}{(n+1)(n+2)}.$$

Q. 25. If $(x^2 + y^2)^2 = xy$, find $\frac{dy}{dx}$.

Ans. We have : $(x^2 + y^2)^2 = xy$,

Diff. w.r.t. x, $2(x^2+y^2).\frac{d}{dx}(x^2+y^2) = x\frac{dy}{dx}+y(1)$

$$\Rightarrow 2(x^2+y^2)\left[2x+2y\frac{dy}{dx}\right] = x\frac{dy}{dx}+y$$

$$\Rightarrow 4x(x^2+y^2) - y = \frac{dy}{dx}[x-4y(x^2+y^2)].$$

Hence, $\frac{dy}{dx} = \frac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$.

SECTION—C

(This section comprises of short answer type questions (SAQ) of 3 marks each)

Q. 26. Find : $\int \frac{\sin\phi}{\sqrt{\sin^2\phi + 2\cos\phi + 3}} d\phi$.

Ans. $I = \int \frac{\sin\phi}{\sqrt{\sin^2\phi+2\cos\phi+3}} d\phi$

$$= \int \frac{\sin\phi}{\sqrt{1-\cos^2\phi+2\cos\phi+3}} d\phi$$

$$= \int \frac{\sin\phi}{\sqrt{4+2\cos\phi-\cos^2\phi}} d\phi.$$

Put $\cos\phi = t$ so that $-\sin\phi\, d\phi = dt$

i.e. $\sin\phi\, d\phi = -dt$.

$$\therefore \quad I = \int \frac{-dt}{\sqrt{4+2t-t^2}}$$

$$= -\int \frac{dt}{\sqrt{-(t^2-2t-4)}}$$

$$= -\int \frac{dt}{\sqrt{-(t^2-2t+1-5)}}$$

$$= -\int \frac{dt}{\sqrt{(5-(t-1)^2}}.$$

Put $t - 1 = z$ so that $dt = dz$.

$$\therefore \quad I = -\int \frac{dz}{\sqrt{(\sqrt{5})^2 - z^2}}$$

$$= \sin^{-1}\frac{z}{\sqrt{5}} + c$$

$$= -\sin^{-1}\frac{t-1}{\sqrt{5}} + c.$$

Hence, $I = -\sin^{-1}\left(\frac{\cos\phi - 1}{\sqrt{5}}\right) + c$.

Q. 27. Two cards are drawn (without replacement) from a well shuffled deck of 52 cards. Find the probability distribution table and mean, variance and standard deviation of number of kings.

Ans. Let 'X' be the random variable, which is the number of kings.

Here, 'X' takes values 0, 1 and 2.

$$\therefore \quad P(X=0) = \frac{48}{52}\times\frac{47}{51} = \frac{188}{221}$$

$$P(X=1) = \frac{4}{52}\times\frac{48}{51} + \frac{48}{52}\times\frac{4}{51} = \frac{32}{221}$$

and $P(X=2) = \frac{4}{52}\times\frac{3}{51} = \frac{1}{221}$.

Hence, the probability distribution is :

X :	0	1	2
P(X) :	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

Now, we have :

X	P(X)	X·P(X)
0	$\frac{188}{221}$	0
1	$\frac{32}{221}$	$\frac{32}{221}$
2	$\frac{1}{221}$	$\frac{2}{221}$
Total		$\frac{34}{221}$

$\therefore$ Mean, $\mu = \Sigma X{\cdot}P(X) = \frac{34}{221} = \frac{2}{13}$.

(*ii*) Variance, $\sigma^2 = \Sigma X^2.P(X) - (\Sigma X.P(X))^2$

$$= \frac{36}{221} - \left(\frac{34}{221}\right)^2 = \frac{6800}{48841} = \frac{400}{2873} = 0.1392$$

(*iii*) S.D., $\sigma = \sqrt{0.1392} = 0.373$.

Q. 28. Evaluate : $\int \sqrt{\sec x - 1}\, dx$.

Ans. Let $I = \int \sqrt{\sec x - 1}\, dx$

$$= \int \sqrt{\frac{1}{\cos x} - 1}\, dx = \int \sqrt{\frac{1 - \cos x}{\cos x}}\, dx$$

$$= \int \sqrt{\frac{2 \sin^2 x/2}{2 \cos^2 x/2 - 1}}\, dx = \int \frac{\sin x/2}{\sqrt{\cos^2 \frac{x}{2} - \frac{1}{2}}}\, dx.$$

Put $\cos \frac{x}{2} = t$ so that $-\sin \frac{x}{2} \cdot \frac{1}{2}\, dx = dt$

i.e. $\sin \frac{x}{2}\, dx = -2\, dt$.

$$\therefore \quad I = \int \frac{-2\, dt}{\sqrt{t^2 - \frac{1}{2}}} = -2 \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

| "**Form :** $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$"

$$= -2 \log \left| t + \sqrt{t^2 - \frac{1}{2}} \right| + c$$

$$= -2 \log \left| \cos \frac{x}{2} + \sqrt{\cos^2 \frac{x}{2} - \frac{1}{2}} \right| + c$$

$$= -2 \log \left| \cos \frac{x}{2} + \frac{1}{\sqrt{2}} \sqrt{2 \cos^2 \frac{x}{2} - 1} \right| + c$$

$$= -2 \log \left| \cos \frac{x}{2} + \frac{1}{\sqrt{2}} \sqrt{\cos x} \right| + c.$$

Or

Evaluate : $\int_0^1 \frac{dx}{e^x + e^{-x}}$.

Ans. Let $I = \int_0^1 \frac{dx}{e^x + e^{-x}}$

$$= \int_0^1 \frac{e^x}{e^{2x} + 1} dx$$

Put $e^x = t$ so that $e^x\, dx = dt$

when $x = 0, t = e^0 = 1$ when $x = 1, t = e^1 = e$.

$$\therefore \quad I = \int_1^e \frac{dt}{\tan^2 + 1}$$

$$= [\tan^{-1} t]_1^e$$

$$= \tan^{-1} e - \tan^{-1}(1) = \tan^{-1} e - \frac{\pi}{4}.$$

Q. 29. Find a particular solution satisfy the given conditions : $x(x^2 - 1)\frac{dy}{dx} = 1$; $y = 0$ when $x = 2$.

Ans. The given differential equation is :

$$x(x^2 - 1)\frac{dy}{dx} = 1.$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x(x^2 - 1)}$$

Integrating, $y = \int \frac{1}{x(x-1)(x+1)} dy$

$$= \int \left(-\frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x-1} + \frac{1}{2} \frac{1}{x+1} \right) dy$$

$$= -\int \frac{1}{x} dy + \frac{1}{2} \int \frac{1}{x-1} dy + \frac{1}{2} \int \frac{1}{x+1} dy$$

$$= -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + c$$

$$= -\log|\sqrt{x^2 - 1}| + \log|x| + c \quad \text{...(1)}$$

when $x = 2$, $y = 0$, $0 = -\log|\sqrt{3}| - \log|2| + c$

$$\Rightarrow \quad c = -\log|2| - \log\sqrt{3}$$

Putting in (1), $y = \log|\sqrt{x^2 - 1}| - \log|x| + \log 2 - \log\sqrt{3}$, which is the reqd. solution.

Or

Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

Ans. We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

By the question, $\frac{dy}{dx} = x + y$

$$\Rightarrow \quad \frac{dy}{dx} - y = x \quad \text{...(1)}$$ | ***Linear Equation***

Comparing with $\frac{dy}{dx} + Py = Q$, we have :

'P' = – 1 and 'Q' = x.

$\therefore$ I.F. $= e^{\int P\, dx} = e^{\int -1 . dx} = e^{-x}$.

Multiplying (1) by e^{-x}, we get :

$$e^{-x} \cdot \frac{dy}{dx} - y \cdot e^{-x} = x.e^{-x}$$

$$\Rightarrow \quad \frac{d}{dx}(y \cdot e^{-x}) = x\, e^{-x}.$$

Integrating, $y.e^{-x} = \int x \cdot e^{-x}\, dx + c$

$$= x \cdot \frac{e^{-x}}{-1} - \int (1) \frac{e^{-x}}{-1} dx + c$$

[*Integrating by parts*]

$$= -x\,e^{-x} + \int e^{-x}\,dx + c$$

$$= -x\,e^{-x} + \frac{e^{-x}}{-1} + c$$

$\Rightarrow \quad y = -x - 1 + c\,e^{x}$ ...(2)

Since the curve passes through the origin (0, 0),

$\therefore \quad 0 = -0 - 1 + c\,e^{0} \quad \Rightarrow \quad c = 1.$

Putting in (2), $y = -x - 1 + e^{x}$

$\Rightarrow \quad x + y + 1 = e^{x},$

which is the reqd. equation of the curve.

Q. 30. Solve the following linear programming problem graphically :

Maximize P = 100*x* + 5*y* subject to the constraints :

$x + y \le 300,$

$3x + y \le 600,$

$y \le x + 200,$

$x, y \ge 0.$

Ans. The system of constraints is :

$x + y \le 300$ ...(1)

$3x + y \le 600$ ...(2)

$y \le x + 200$ ...(3)

and $x, y \ge 0$ ...(4)

The shaded portion in the following figure is the feasible region determined by the system of constraints (1) – (4).

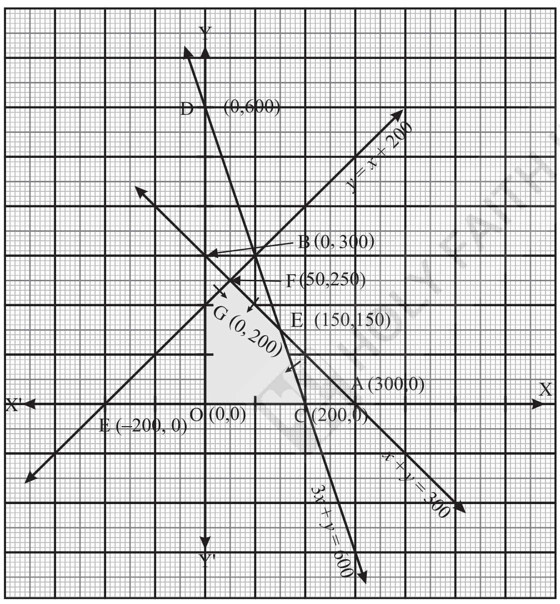


Fig.

It is observed that the feasible region OCEFGO is bounded.

Thus, we use ***Corner Point Method*** to determine the maximum value of Z, where Z = 100*x* + 5*y*.

The corner points are O (0, 0), C (200, 0),

E (150, 150), [*Solving 3x + y = 600 and x + y = 300*]

F (50, 250) [*Solving 3x + y = 600 and y = x + 200*]

and G (0, 200) respectively.

We evaluate Z at each corner point :

Corner Point	Corresponding Value of P = 100x + 5y
O : (0, 0)	0
C : (200, 0)	**20000 (Maximum)**
E : (150, 150)	15750
F : (50, 250)	6250
G : (0, 200)	1000

Hence, Z_{max} = 20000 at (200,0).

Or

Solve graphically the following linear programming problem :

Maximise : Z = 6*x* + 3*y*, subject to the constraints :

$4x + y \ge 80,$

$3x + 2y \le 150,$

$x + 5y \ge 115,$

$x \ge 0, y \ge 0.$

Ans. The system of constraints is :

$4x + y \ge 80$ ...(1)

$3x + 2y \le 150$ ...(2)

$x + 5y \ge 115$ ...(3)

and $x \ge 0, y \ge 0$ ...(4)

The shaded portion in the following figure is the feasible region determined by the system of constraints (1) – (4).

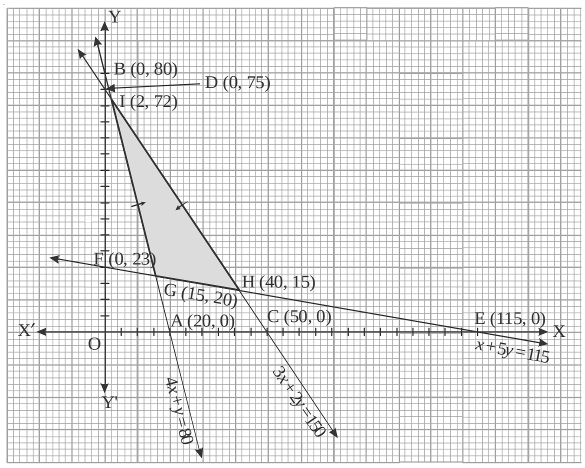


Fig.

It is observed that the feasible region GHI is bounded.

Thus, we use ***Corner Point Method*** to determine the maximum value of Z, when Z = 6*x* + 3*y*.

The corner points are G(15, 20), H (40, 15) and I (2, 72).

We evaluate Z at each corner point.

Corner point	Corresponding Value of Z
G : (15, 20)	150
H : (40, 15)	**285 (Maximum)**
I : (2, 72)	228

Hence, Z_{max} = 285 at (40, 15).

Q. 31. Maximise and Minimise :
$Z = 4x + 3y - 7$
subject to the constraints :
$x + y \leq 10, x + y \geq 3, x \leq 8, y \leq 9, x, y \geq 0.$

Ans. The given system of constraints is :

$x + y \leq 10$ ...(1)
$x + y \geq 3$ ...(2)
$x \leq 8$ ...(3)
$y \leq 9$ ...(4)
$x, y \geq 0$ ...(5)

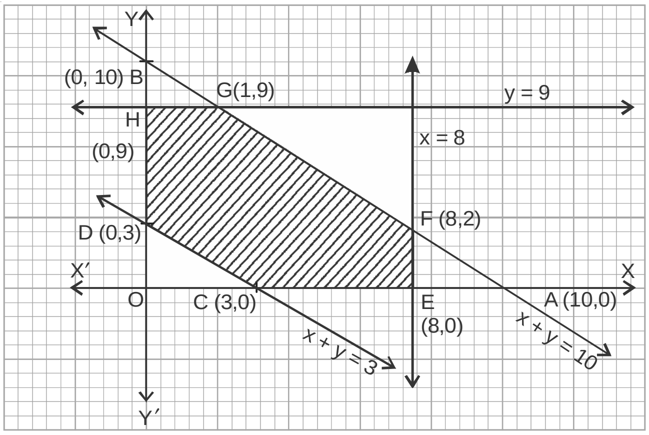


Fig.

The shaded region in the above figure is the feasible region determined by the system of constraints (1) – (5). It is observed that the feasible region DCEFGH is bounded. Thus we use ***Corner Point Method*** to determine the maximum and minimum values of Z, where :

$Z = 4x + 3y - 7$ ...(6)

The co-ordinates of D, C, E, F, G and H are respectively (0, 3), (3, 0), (8, 0), (8, 2),
[*Solving x = 8 and x + y = 10*]
(1, 9) [*Solving y = 9 and x + y = 10*]
and (0, 9).

We evaluate Z at each corner point:

Corner Point	Corresponding Value of Z
D : (0, 3)	**2 (Minimum)**
C : (3, 0)	5
E : (8, 0)	25
F : (8, 2)	**31 (Maximum)**
G : (1, 9)	24
H : (0, 9)	20

Hence, $Z_{max} = 31$ at (8, 2) and $Z_{min} = 2$ at (0, 3).

SECTION—D

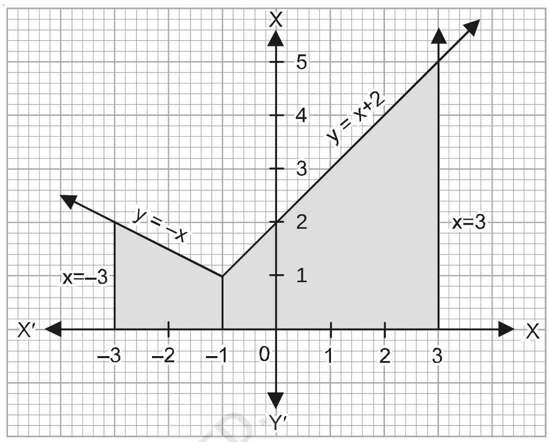
(This section comprsies of Long Answer Type questions (LAQ) of 5 marks each)

Q. 32. Draw a rough sketch of the curve :
$y = 1 + |x + 1|, x = -3, x = 3, y = 0$ and find the area of the region bounded by them, using integration.

Ans. Graph : We have :

$$y = 1 + |x + 1|$$
$$= 1 + x + 1 \quad \text{if} \quad x \geq -1$$
$$= x + 2 \quad \text{if} \quad x \geq -1$$
and
$$= 1 - (x + 1) \quad \text{if} \quad x < -1$$
$$= -x \quad \text{if} \quad x < -1.$$

The graph is as shown below :



$$\text{Required area} = \int_{-3}^{-1} -x\,dx + \int_{-1}^{3} (x+2)\,dx$$

$$= \left[-\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$$

$$= -\frac{1}{2}(1-9) + \left[\left(\frac{9}{2}+6\right) - \left(\frac{1}{2}-2\right)\right]$$

$$= 4 + (4 + 8) = 4 + 12 = 16 \text{ sq. unit.}$$

Q. 33. Show that the relation R in the set A of points in a plane given by R = {(P, Q) : distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all points related to a point P ≠ (0, 0) is the circle passing through P with origin as centre.

Ans. Here R = {(P, Q) : | OP | = | OQ |, where O is the origin}

R is reflexive. [∵ | *OP* | = | *OP* |,
∴ *(P, P)* ∈ *R* ∀ *P* ∈ *A*]

R is symmetric. [∵ *(P, Q)* ∈ *R* ⇒ | *OP* | = | *OQ* |]
⇒ [| *OQ* | = | *OP* | ⇒ *(Q, P)* ∈ *R*]

R is transitive. [∵ *(P, Q)* ∈ *R and (Q, S)* ∈ *R*
⇒ | *OP* | = |*OQ* | *and* | *OQ* | = | *OS* |
⇒ | *OP* | = | *OS* | ⇒ *(P, S)* ∈ *R*]

Hence, R is an equivalence relation.

Set of points related to P ≠ (0, 0)
= {Q ∈ A : (Q, P) ∈ R}
= {Q ∈ A : | OQ | = | OP |}
= {Q ∈ A : Q lies on the circle through P with centre O}.

Or

Let $f : N \to N$ be defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, \text{ if } n \text{ is odd} \\ \dfrac{n}{2}, \text{ if } n \text{ is even} \end{cases}$ **for all $n \in N$.**

State whether the function f is bijective. Justify your answer.

Ans. Here $f(1) = \frac{1+1}{2} = 1$,

$f(2) = \frac{2}{2} = 1$

$f(3) = \frac{3+1}{2} = 2$, $f(4) = \frac{4}{2} = 2$.

Thus $f(2k-1) = \frac{(2k-1)+1}{2} = k$

and $f(2k) = \frac{2k}{2} = k$

$\Rightarrow f(2k-1) = f(2k)$, where $k \in \mathbf{N}$

$\Rightarrow$ f is not one-one.

But f is onto because $R_f = \mathbf{N}$

[$\because$ *For any $x \in \mathbf{N}$, $2x \in \mathbf{N}$ such that $f(2x) = \frac{2x}{2} = x$*]

$\Rightarrow$ f is onto.

Hence, 'f' is not bijective.

Q. 34. Show that the points A(1, 2, 7), B(2, 6, 3) and C (3, 10, –1) are collinear.

Ans. If O be the origin,

then $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 7\hat{k}$,

$\overrightarrow{OB} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

and $\overrightarrow{OC} = 3\hat{i} + 10\hat{j} - \hat{k}$.

$\therefore$ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (2\hat{i} + 6\hat{j} + 3\hat{k}) - (\hat{i} + 2\hat{j} + 7\hat{k})$

$= \hat{i} + 4\hat{j} - 4\hat{k}$,

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (3\hat{i} + 10\hat{j} - \hat{k}) - (2\hat{i} + 6\hat{j} + 3\hat{k})$

$= \hat{i} + 4\hat{j} - 4\hat{k}$

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (3\hat{i} + 10\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 7\hat{k})$

$= 2\hat{i} + 8\hat{j} - 8\hat{k}$

$= 2(\hat{i} + 4\hat{j} - 4\hat{k})$.

$\therefore$ AB $= |\overrightarrow{AB}| = \sqrt{1^2 + 4^2 + (-4)^2}$

$= \sqrt{1+16+16} = \sqrt{33}$,

BC $= |\overrightarrow{BC}| = \sqrt{1^2 + 4^2 + (-4)^2}$

$= \sqrt{1+16+16} = \sqrt{33}$

and AC $= |\overrightarrow{AC}| = \sqrt{2^2 + 8^2 + (-8)^2}$

$= \sqrt{4+64+64} = \sqrt{132}$

$= 2\sqrt{33}$.

Clearly AB + BC = AC.

Hence, A, B, C are collinear.

Or

Find the shortest distance and the vector equation of the line of shortest distance between the lines given by :

$\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + l\,(3\hat{i} - \hat{j} + \hat{k})$ **and**

$\vec{r} = (-3\hat{i} - 7\hat{j} + 6\hat{k}) + \mu\,(-3\hat{i} + 2\hat{j} + 4\hat{k})$.

Ans. The given equations in the cartesian form are :

$L_1 : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} (= \lambda)$ ...(1)

and $L_2 : \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} (= \mu)$ ...(2)

Any point on L_1 is $(3\lambda + 3, -\lambda + 8, \lambda + 3)$.

Any point on L_2 is $(-3\mu - 3, 2\mu - 7, 4\mu + 6)$.

If the line of shortest distance intersects (1) in P and (2) in Q, then the direction-ratios of $\overrightarrow{PQ}$ are :

$< -3\mu - 3 - 3\lambda - 3, 2\mu - 7 + \lambda - 8, 4\mu + 6 - \lambda - 3 >$

i.e. $< 3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3 >$.

Since PQ is perp. to line (1),

$\therefore$ $(3)(3\lambda + 3\mu + 6) + (-1)(-\lambda - 2\mu + 15) + (1)(\lambda - 4\mu - 3) = 0$

$\Rightarrow$ $11\lambda + 7\mu = 0$ ...(3)

Since PQ is perp. to line (2),

$\therefore$ $(-3)(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$

$\Rightarrow$ $-7\lambda - 29\mu = 0$ ...(4)

Solving (3) and (4), $\lambda = 0$, $\mu = 0$.

$\therefore$ Points P and Q are (3, 8, 3) and (– 3, – 7, 6) respectively.

$\therefore$ S.D. = | PQ |

$= \sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2}$

$= \sqrt{36 + 225 + 9} = \sqrt{270} = 3\sqrt{30}$ units

and the vector equation of line of S.D. is :

$\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \mu\,(-6\hat{i} - 15\hat{j} + 3\hat{k})$.

[*Using* $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$]

Q. 35. Two numbers are selected at random (without replacement) from positive numbers 2, 3, 4, 5, 6 and 7. Let X denote the larger of the two numbers obtained. Find the probability distribution of X.

Ans. Here, $n(S) = {}^6C_2 = \frac{6\times5}{2\times1} = 15$,

Let X be the larger of that two numbers obtained.

$\therefore$ X = 3, 4, 5, 6, 7.

$\therefore$ Probability distribution is :

X :	3	4	5	6	7
P(X) :	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

SECTION—E

(This section comprsies of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36. Three shopkeepers P, Q and R go to a store to buy stationary. P purchases 12 dozen notebooks, 5 dozen pens and 6 dozen pencils. Q purchases 10 dozen notebooks, 6 dozen pens and 7 dozen pencils. R purchases 11 dozen notebooks, 13 dozen pens and 8 dozen pencils. A notebook costs ₹40, a pen costs ₹12 and a pencil costs ₹3.

Based on above information, answer the following :

(*a*) Find the number of items purchased by shopkeepers P, Q and R represented in matrix form.

(*b*) If Y represents the matrix formed by the cost of each item and X represents the matrix formed by the number of items, then find XY.

Ans. (*a*) $X = \begin{bmatrix} \text{Notebooks} & \text{Pens} & \text{Pencils} \\ 12 & 5 & 6 \\ 10 & 6 & 7 \\ 11 & 13 & 8 \end{bmatrix}$

(*b*) $Y = \begin{bmatrix} 40\times12 \\ 12\times12 \\ 3\times12 \end{bmatrix} = \begin{bmatrix} 480 \\ 144 \\ 36 \end{bmatrix}$

$\therefore \quad XY = \begin{bmatrix} 12\times480+5\times144+6\times36 \\ 10\times480+6\times144+7\times36 \\ 11\times480+13\times144+8\times36 \end{bmatrix}$

$= \begin{bmatrix} 5760+720+216 \\ 4800+864+252 \\ 5280+1872+288 \end{bmatrix} = \begin{bmatrix} 6696 \\ 5916 \\ 7440 \end{bmatrix}.$

Hence, X = ₹ 6696, Y = ₹ 5916, Z = ₹ 7440.0

Q. 37. Answer the questions based on the given information :

Two metal rods, R_1 and R_2, of lengths 1 cm and 12 cm respectively, are insulated at both ends. Rod R_1 is being heated from a specific point while rod R_2 is being cooled from a specific point. The temperature (T) in Celsius within both rods fluctuates based on the distance (*x*) measured from either end. The temperature at a particular point along the rod is determined by the equation :

T = (16 – *x*) *x* and T (*x* – 12) *x* for rods R_1 and R_2 respectively, where the distance *x* is measured in metres from one of the ends.

(*a*) Find the rate of change of temperature at the midpoint of the rod that is being heated. Show your steps.

(*b*) Find the maximum temperature attained by the rod that is being cooled. Show your steps.

Ans. (*a*) We have : T = (16 – *x*) *x* and T = (*x* – 12) *x*.

$\therefore \quad \frac{dT}{dx} = 16 - 2x \quad \text{and} \quad \frac{dT}{dx} = 2x - 12$

$\therefore \quad \left.\frac{dT}{dx}\right]_{x=8} = 16 - 16 = 0 \text{ and } \left.\frac{dT}{dx}\right]_{x=8}$

$= 12 - 12 = 0$

(*b*) $\left.T\right]_{x=8} = (16-8)\,8 = 8\times8 = 64$

and $\left.T\right]_{x=6} = (6-12)\,6 = -36.$

Q. 38. Milie purchased an air plant holder, which is in the shape of a tetrahedron.

Let A, B, C and D are the coordinates of the air plant holder where A ≡ (1, 1, 1), B ≡ (2, 1, 3), C ≡ (3, 2, 2) and D ≡ (3, 3, 4).

Based on the above information, answer the following questions.

(I) Find the position vector of $\overrightarrow{AB}$.

(II) Find the position vector of $\overrightarrow{AC}$.

(III) Find the position vector of $\overrightarrow{AD}$.

Or

Find the area of ΔABC.

Ans. (I) P.V. of $\overrightarrow{AB} = (2\hat{i}+\hat{j}+3\hat{k}) - (\hat{i}+\hat{j}+\hat{k})$

$= \hat{i}+2\hat{k}.$

(II) P.V. of $\overrightarrow{AC} = (3\hat{i}+2\hat{j}+2\hat{k}) - (\hat{i}+\hat{j}+\hat{k})$

$= 2\hat{i}+\hat{j}+\hat{k}.$

(III) P.V. of $\overrightarrow{AD} = (3\hat{i}+3\hat{j}+4\hat{k}) - (\hat{i}+\hat{j}+\hat{k})$

$= 2\hat{i}+2\hat{j}+3\hat{k}.$

Or

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix} = -2\hat{i}+3\hat{j}+\hat{k}.$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$= \frac{1}{2}\sqrt{4+9+1} = \frac{\sqrt{14}}{2}$ sq. units.

Holy Faith New Style Sample Paper–8

(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions : Same as Holy Faith New Style Sample Paper–1.

SECTION—A

(Multiple choice questions, each question carries 1 mark)

1. Let R be the relation in the set **N** given by : R = {(*a*, *b*) : $a = b - 2$, $b > 6$}. Then :

(*a*) (2, 4) ∈ R (*b*) (3, 8) ∈ R
(*c*) (6, 8) ∈ R (*d*) (8, 7) ∈ R.

Ans. (*c*) (6, 8) ∈ R.

2. If $\tan^{-1} x = y$, then :

(*a*) $-1 < y < 1$ (*b*) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(*c*) $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (*d*) $y \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$.

Ans. (*c*) $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

3. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ is equal to :

(*a*) $\frac{\pi}{5}$ (*b*) $\frac{2\pi}{3}$

(*c*) $\frac{\pi}{2}$ (*d*) $\frac{\pi}{3}$.

Ans. (*b*) $\frac{2\pi}{3}$.

4. $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to :

(*a*) $\frac{x}{\sqrt{1-x^2}}$ (*b*) $\frac{1}{1-x^2}$

(*c*) $\frac{1}{\sqrt{1+x^2}}$ (*d*) $\frac{x}{\sqrt{1+x^2}}$.

Ans. (*d*) $\frac{x}{\sqrt{1+x^2}}$.

5. The value of the determinant $\begin{vmatrix} 6 & 0 & -1 \\ 2 & 1 & 4 \\ 1 & 1 & 3 \end{vmatrix}$ is :

(*a*) 10 (*b*) 8
(*c*) 7 (*d*) –7.

Ans. (*d*) –7.

6. A function $f : \mathbf{R} \to \mathbf{R}$ is defined by :

$$f(x) = \begin{cases} e^{-2x}, & x < l_n\frac{1}{2} \\ 4, & l_n\frac{1}{2} \le x \le 0 \\ e^{-2x}, & x > 0. \end{cases}$$

Which of the following statements is true about the function at the point $x = l_n\frac{1}{2}$?

(*a*) $f(x)$ is not continuous but differentiable
(*b*) $f(x)$ is continuous but not differentiable
(*c*) $f(x)$ is neither continuous nor differentiable
(*d*) $f(x)$ is both continuous as well as differentiable.

Ans. (*b*) $f(x)$ is continuous but not differentiable.

7. If $(x^2 + y^2)^2 = xy$, then $\frac{dy}{dx}$ is :

(*a*) $\frac{y+4x(x^2+y^2)}{4y(x^2+y^2)-x}$ (*b*) $\frac{y-4x(x^2+y^2)}{x+4(x^2+y^2)}$

(*c*) $\frac{y-4x(x^2+y^2)}{4y(x^2+y^2)-x}$ (*d*) $\frac{4y(x^2+y^2)-x}{y-4x(x^2+y^2)}$.

Ans. (*c*) $\frac{y-4x(x^2+y^2)}{4y(x^2+y^2)-x}$.

8. The absolute maximum value of the function $f(x) = 4x - \frac{1}{2}x^2$ in the interval $\left[-2, \frac{9}{2}\right]$ is :

(*a*) 8 (*b*) 9
(*c*) 6 (*d*) 10.

Ans. (*a*) 8.

9. $\int \frac{1}{x(x^2+1)} dx$

(*a*) $\log|x| - \frac{1}{2}\log(x^2+1) + c$

(*b*) $\log|x| + \frac{1}{2}\log(x^2+1) + c$

(*c*) $\log|x| - \frac{1}{2}\log(x^2-1) + c$

(*d*) $\log|x| + \frac{1}{2}\log(x^2-1) + c$.

Ans. (*a*) $\log|x| - \frac{1}{2}\log(x^2+1) + c$.

10. $\int_0^{\pi/8} \tan^2(2x)\,dx$ is equal to :

(*a*) $\frac{4-\pi}{8}$ (*b*) $\frac{4+\pi}{8}$

(*c*) $\frac{4-\pi}{4}$ (*d*) $\frac{4-\pi}{2}$.

Ans. (*a*) $\frac{4-\pi}{8}$.

11. Area of the region bounded by the curve $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant is :

(*a*) $8(4-\sqrt{2})$ (*b*) $\frac{8}{3}(4-\sqrt{2})$

(*c*) $\frac{3}{8}(4-\sqrt{2})$ (*d*) $4-\sqrt{2}$.

Ans. (*b*) $\frac{8}{3}(4-\sqrt{2})$.

12. Which of the following differential equations cannot be solved, using variable separable method :

(*a*) $\frac{dy}{dx} = e^{x+y} + e^{-x+y}$

(*b*) $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$

(*c*) $xy\frac{dy}{dx} = 1 + x + y + xy$

(*d*) $\frac{dy}{dx} + y = 2$.

Ans. (*b*) $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$.

13. The order and degree of the differential equation :

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = \frac{d^2y}{dx^2}$$ respectively, are :

(*a*) 1, 2 (*b*) 2, 2

(*c*) 2, 1 (*d*) 4, 2.

Ans. (*c*) 2, 1.

14. A unit vector $\hat{a}$ makes equal but acute angles on the coordinate axes. The projection of the vector $\hat{a}$ on the vector $\vec{b} = 5\hat{i} + 7\hat{j} - \hat{k}$ is :

(*a*) $\frac{11}{15}$ (*b*) $\frac{11}{5\sqrt{3}}$

(*c*) $\frac{4}{5}$ (*d*) $\frac{3}{5\sqrt{3}}$.

Ans. (*a*) $\frac{11}{15}$.

15. If $\vec{a}\cdot\vec{b} = \frac{1}{2}|\vec{a}||\vec{b}|$, then the angle between $\vec{a}$ and $\vec{b}$ is :

(*a*) 0° (*b*) 30°

(*c*) 60° (*d*) 90°.

Ans. (*c*) 60°.

16. Direction-cosines of the line :

$\frac{x-1}{2} = \frac{1-y}{3} = \frac{2z-1}{12}$ are :

(*a*) $<\frac{2}{7}, \frac{3}{7}, \frac{6}{7}>$ (*b*) $<\frac{2}{\sqrt{157}}, \frac{-3}{\sqrt{157}}, \frac{12}{\sqrt{157}}>$

(*c*) $<\frac{2}{7}, -\frac{3}{7}, -\frac{6}{7}>$ (*d*) $<\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}>$.

Ans. (*d*) $<\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}>$.

17. The corner points of the feasible region for a Linear Programming problem are P (0, 5), Q (1, 5), R (4, 2) and S (12, 0). The minimum value of the objective function Z = 2*x* + 5*y* is at the point :

(*a*) P (*b*) Q

(*c*) R (*d*) S.

Ans. (*c*) R.

18. If A and B are two events such that P(A) = 0.2, P(B) = 0.4 and P(A ∪ B) = 0.5, then value of P (A/B) is:

(*a*) 0.1 (*b*) 0.25

(*c*) 0.5 (*d*) 0.08.

Ans. (*c*) 0.5.

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.

(*b*) Both 'A' and 'R' are true but 'R' is not the correct explanation of 'A'.

(*c*) 'A' is true but 'R' is false.

(*d*) 'A' is false but 'R' is true.

19. Statement–(A) :

If y satisfies $x^2\frac{dy}{dx} + xy = \sin x$, $y(1) = 2$, then the value of $y(2)$ is $\frac{1}{2}\int_1^2 \frac{\sin t}{t}\,dt$.

Statement–(R) :

The solution of the linear equation $\frac{dy}{dx} + Py = Q$ can be obtained by multiplying with the factor $e^{\int P\,dx}$.

Ans. (*d*)'A' is false but 'R' is true.

20. Statement–A : The point A (1, 0, 7) is the mirror image of the point B (1, 6, 3) in the line : $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

Statement–R : The line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ bisects the line-segment joining A (1, 0, 7) and B (1, 6, 3).

Ans. Mid-point of AB is M (1, 3, 5).

This lies on $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ ...(1)

Now direction-ratios of AB are :
< 1 – 1, 0 – 6, 7 – 3 > *i.e.* < 0, 6, – 4 >
and direction-ratios of (1) are < 1, 2, 3 >.
Since (1) (0) + (2) (6) + (3) (– 4) = 0,
∴ AB is perp. to the given line (1).
Hence, (B) is true.

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Write in simplest form : $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x<\pi$.

Ans. LHS = $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

$$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$\left[\because 1-\cos 2\theta = 2\sin^2\theta;\ 1+\cos 2\theta = 2\cos^2\theta\right]$$

$$= \tan^{-1}\left(\sqrt{\tan^2\frac{x}{2}}\right) = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}.$$

Or

Write in simplest form :

$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), a>0; \frac{-a}{\sqrt{3}} \le x \le \frac{a}{\sqrt{3}}$.

Ans. $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$

$$= \tan^{-1}\left(\frac{3\frac{x}{a}-\frac{x^3}{a^3}}{1-\frac{3x^2}{a^2}}\right).$$

Put $\frac{x}{a} = \tan\theta$ so that $\theta = \tan^{-1}\frac{x}{a}$.

∴ Given expression

$$= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta)$$

$$= 3\theta = 3\tan^{-1}\frac{x}{a}.$$

Q. 22.A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing ?

Ans. Let 'r' be the radius of the circular wave.

Then $A = \pi r^2$,

where A is the enclosed area at time t.

Differentiating w.r.t. t, we have :

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi r\,(5) \qquad \left[\because \frac{dr}{dt} = 5\ cm/s\right]$$

$$= 10\pi r.$$

When r = 8 cm, $\frac{dA}{dt} = 10\pi\,(8) = 80\pi$ cm^2/sec.

Hence, the enclosed area is increasing at the rate of 80π cm^2/s, where r = 8 cm.

Q. 23. Prove that the function $g(x)$ = log x do not have maxima or minima.

Ans. We have $f(x)$ = log x.

∴ $f'(x) = \frac{1}{x}$.

Now $f'(x) = 0$

⇒ $\frac{1}{x} = 0$,

which gives no real value of x.

Hence, $f(x)$ does not have maximum or minimum values.

Or

Calculate the absolute maximum and absolute minimum value of the function f (x) = $\frac{x+1}{\sqrt{x^2+1}}$, 0 ≤ x ≤ 2.

Ans. We have : $f(x) = \frac{x+1}{\sqrt{x^2+1}}$.

$$\therefore f'(x) = \frac{\sqrt{x^2+1}\,(1+0)-(x+1)\frac{1}{2\sqrt{x^2+1}}(2x+0)}{(x^2+1)}$$

$$= \frac{\sqrt{x^2+1}-\frac{x^2+x}{\sqrt{x^2+1}}}{x^2+1} = \frac{x^2+1-x^2-x}{(x^2+1)^{3/2}} = \frac{1-x}{(x^2+1)^{3/2}}.$$

Now $f'(x) = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1 \in [0, 2]$.

Now $f(0) = \frac{0+1}{\sqrt{0+1}} = \frac{1}{1} = 1$,

$f(1) = \frac{1+1}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

and $f(2) = \frac{2+1}{\sqrt{4+1}} = \frac{3}{\sqrt{5}}$.

Hence, absolute maximum value is $\frac{3}{\sqrt{5}}$ and absolute min. value is 1.

Q. 24. Show that $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)\,dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$.

Ans. Let $I = \int_0^a f(x)\,g(x)\,dx$

$$= \int_0^a f(a-x)\,g(a-x)\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

But $f(a-x) = f(x)$

and $g(x) + g(a-x) = 4$ [*Given*]

$\Rightarrow \quad g(a-x) = 4 - g(x)$.

$$\therefore I = \int_0^a f(x)\left[4 - g(x)\right]dx = 4\int_0^a f(x)\,dx - I$$

$$\Rightarrow 2I = 4\int_0^a f(x)\,dx \Rightarrow I = 2\int_0^a f(x)\,dx.$$

Q. 25. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.

Ans. Let $y = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$$= \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right)$$

$$= \frac{\pi}{2} - \frac{x}{2}.$$

Hence, $\frac{dy}{dx} = 0 - \frac{1}{2} = -\frac{1}{2}$.

SECTION—C

(This section comprises of short answer type questions (SAQ) of 3 marks each)

Q. 26. Evaluate : $\int \frac{x^2-1}{x^4+1}dx$.

Ans. $I = \int \frac{x^2-1}{x^4+1}dx = \int \frac{\left(1 - \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2}}dx$.

[*Dividing Num. and Denom. by x^2*]

Put $x + \frac{1}{x} = t$ so that $\left(1 - \frac{1}{x^2}\right)dx = dt$.

Also $x^2 + 2 + \frac{1}{x^2} = t^2 \quad \Rightarrow x^2 + \frac{1}{x^2} = t^2 - 2$.

$$\therefore I = \int \frac{dt}{t^2-2} = \int \frac{dt}{t^2 - (\sqrt{2})^2} \quad \Big| \text{"Form}: \int \frac{1}{x^2-a^2}\text{"}$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + c$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}}\right| + c$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}\right| + c.$$

Q. 27. If $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $(P(A \cap B) = \frac{4}{13}$, evaluate $P(A/B)$.

Ans. $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{4/13}{9/13} = \frac{4}{9}$.

Q. 28. Evaluate : $\int \sqrt{\tan x}\,dx$.

Ans. Let $I = \int \sqrt{\tan x}\,dx$.

Put $\sqrt{\tan x} = t$ *i.e.* $\tan x = t^2$

so that $\sec^2 x\,dx = 2t\,dt$ *i.e.* $dx = \frac{2t\,dt}{1+\tan^2 x} = \frac{2t\,dt}{1+t^4}$.

$$\therefore \quad I = \int t\frac{2t}{1+t^4}dt = \int \frac{2t^2}{1+t^4}dt$$

$$= \int \frac{(t^2+1)+(t^2-1)}{t^4+1}dt$$

(*Note this step*)

$$= \int \frac{t^2+1}{t^4+1}dt + \int \frac{t^2-1}{t^4+1}dt$$

$$= I_1 + I_2 \quad \text{....(1) (say)}$$

Now $\quad I_1 = \int \frac{t^2+1}{t^4+1}dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt$

[*Dividing Num. & Denom. by t^2*]

Put $\mathbf{t - \frac{1}{t} = y}$ so that $\left(1+\frac{1}{t^2}\right)dt = dy$.

Also $\quad t^2 + \frac{1}{t^2} = y^2 + 2.$

$\therefore \quad I_1 = \int \frac{dy}{y^2+2} = \int \frac{dy}{(\sqrt{2})^2 + y^2}$

| "**Form:** $\int \frac{dx}{a^2+x^2}$"

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right).$$

And, $\quad I_2 = \int \frac{t^2-1}{t^4+1}dt = \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt$

[*Dividing Num. & Denom. by t^2*]

Put $\mathbf{t + \frac{1}{t} = z}$ so that $\left(1-\frac{1}{t^2}\right)dt = dz.$

Also $\quad t^2 + \frac{1}{t^2} = z^2 - 2.$

$\therefore \quad I_2 = \int \frac{dz}{z^2-2} = \int \frac{dz}{z^2 - (\sqrt{2})^2}$

| "**Form:** $\int \frac{dx}{x^2-a^2}$"

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right|$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right|.$$

$\therefore$ From (1), $I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)$

$$+ \frac{1}{2\sqrt{2}}\log\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right| + c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right)$$

$$+ \frac{1}{2\sqrt{2}}\log\left|\frac{\tan x - \sqrt{2\tan x}+1}{\tan x + \sqrt{2\tan x}+1}\right| + c.$$

Or

Evaluate : $\displaystyle\int_0^{\pi/2} \mathbf{\log \sin x\, dx}$ **.**

Ans. Let $\displaystyle I = \int_0^{\pi/2} \log \sin x\, dx \quad \text{...(1)}$

$\therefore \quad \displaystyle I = \int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2}-x\right)\right]dx$ [*Property V*]

$\Rightarrow \quad \displaystyle I = \int_0^{\pi/2} \log\ \cos x\, dx \quad \text{...(2)}$

Adding (1) and (2),

$$2I = \int_0^{\pi/2} \log\ \sin x\, dx + \int_0^{\pi/2} \log \cos x\, dx$$

$$= \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$= \int_0^{\pi/2} \log \sin x \cos x\, dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log(\sin 2x - \log 2)\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx \quad \text{...(3)}$$

For $\displaystyle\int_0^{\pi/2} \log \sin 2x\, dx$ *:*

Put 2x = t so that $2dx = dt$ i.e. $dx = \frac{1}{2}\, dt$.

When $x = 0, t = 0$. When $x = \frac{\pi}{2}, t = \pi$.

$$\therefore \int_0^{\pi/2} \log \sin 2x\, dx = \int_0^{\pi} \log \sin t \cdot \frac{1}{2}\, dt$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin x\, dx \qquad [Property\ I]$$

$$= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \log \sin x\, dx$$

$$[\because \log \sin(\pi - x) = \log \sin x]$$

$$= \int_0^{\pi/2} \log \sin x\, dx = I.$$

$\therefore$ From (3), $2I = I - \log 2 \cdot [x]_0^{\pi/2}$

$$\Rightarrow \quad I = -\log 2 \cdot \left[\frac{\pi}{2} - 0\right].$$

Hence, $I = -\frac{\pi}{2}\log 2$.

Q. 29. For the differential equation $xy\frac{dy}{dx} = (x+2)(y+2)$, find the solution curve passing through the point (1, –1).

Ans. The given equation is :

$$xy\frac{dy}{dx} = (x+2)(y+2)$$

$$\Rightarrow \quad \frac{y\,dy}{y+2} = \frac{x+2}{x}\,dx. \qquad |\textit{Variables Separable}$$

Integrating, $\int \frac{y}{y+2}\,dy = \int \frac{x+2}{x}\,dx + c$

$$\Rightarrow \quad \int \frac{\overline{y+2} - 2}{y+2}\,dy = \int \left(1 + \frac{2}{x}\right)dx + c$$

$$\Rightarrow \quad \int 1 \cdot dy - 2\int \frac{1}{y+2}\,dy = \int 1 \cdot dx + 2\int \frac{1}{x}\,dx + c$$

$$\Rightarrow \quad y - 2\log|y+2| = x + 2\log|x| + c \qquad ...(1)$$

which represents the family of curves.

The required curve passes through (1, – 1).

$\therefore \quad -1 - 2\log|1| = 1 + 2\log|1| + c$

$\Rightarrow \quad -1 - 2(0) = 1 + 2(0) + c$

$[\because \log|1| = \log 1 = 0]$

$\Rightarrow \quad -1 = 1 + c \quad \Rightarrow \quad c = -2.$

Putting in (1), $y - 2\log|y+2| = x + 2\log|x| - 2$

$\Rightarrow \quad y - x + 2 = \log|x^2(y+2)^2|$,

which is the reqd. solution curve.

Or

Find the particular solution satisfly the given condition :

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0;\ y = \frac{\pi}{4} \text{ when } x = 1.$$

Ans. The given differential equation is :

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$$

$$\Rightarrow \quad \left[\sin^2\left(\frac{y}{x}\right) - \frac{y}{x}\right] + \frac{dy}{dv} = 0 \qquad ...(1)$$

Put $\frac{y}{x} = v$ i.e., $y = vx$

so that $\frac{dy}{dv} = v + x$

$\therefore$ (1) becomes : $(\sin^2 v - v) + v + = 0$

$$\Rightarrow \quad \sin^{-2} v + x\frac{dy}{dv} = 0$$

$$\Rightarrow \quad \frac{dx}{x} + \text{cosec}^2 dv = 0 \qquad |\textit{Variables Separable}$$

Integrating, $\int \frac{dx}{x} + \int \text{cosec}^2 v\,dv = 0$

$\Rightarrow \log|x| + (-\cot v) = 0$

$\Rightarrow \log|x| - \cot\frac{y}{x} = 0 \qquad ...(2)$

When $x = 1, y = \frac{\pi}{4}$

$\therefore \log(1) - \cot\frac{\pi/4}{1} = 0$

$\Rightarrow \quad 0 - 1 = 0$

Put in (2), $\log|x| - \cot\frac{y}{x} = -1$

$\Rightarrow \log|x| - \cot\frac{y}{x} + 1 = 0,$

which is the reqd. particular solution.

Q. 30. Maximise Z = 3*x* + 4*y*

subject to the constraints : $x + y \le 4$, $x \ge 0$, $y \ge 0$.

Ans. The system of constraints is :

$$x + y \le 4 \quad \text{...(1)}$$

$$\text{and } x \ge 0,\ y \ge 0 \quad \text{...(2)}$$

The shaded region in the following figure is the feasible region determined by the system of constraints (1) – (2)

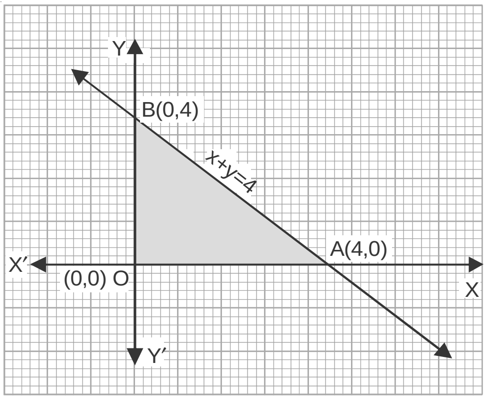


Fig.

It is observed that the feasible region OAB is bounded. Thus we use ***Corner Point Method*** to determine the maximum value of Z, where :

$$Z = 3x + 4y \quad \text{...(3)}$$

The co-ordinates of O, A and B are (0, 0), (4, 0) and (0, 4) respectively.

We evaluate Z at each corner point :

Corner Point	Corresponding value of Z
O : (0, 0)	0
A : (4, 0)	12
B : (0, 4)	**16 (Maximum)**

Hence, $Z_{max} = 16$ at the point (0, 4).

Or

If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ **find A^{-1}. Hence, solve the system of equations:**

$2x + 3y + 4z = 17$, $x - y = 3$, $y + 2z = 7$.

Ans. (I) $|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{vmatrix}$

$$= 2(-2-0) - 1(6-4)$$

$$= -4 - 2 = -6 \neq 0.$$

$\therefore$ A is non-singular and as such A^{-1} exists.

Now $A_{11} = \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} = -2 - 0 = -2;$

$$A_{12} = -\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -(2-0) = -2;$$

$$A_{13} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 + 0 = 1.$$

$$A_{21} = -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -(6-4) = -2;$$

$$A_{22} = \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4;$$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = -(2-0) = -2.$$

$$A_{31} = \begin{vmatrix} 3 & 4 \\ -1 & 0 \end{vmatrix} = 0 + 4 = 4;$$

$$A_{32} = -\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = -(0-4) = 4;$$

$$A_{33} = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -2 - 3 = -5.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}'$$

$$= \begin{bmatrix} -2 & -2 & 1 \\ -2 & 4 & -2 \\ 4 & 4 & -5 \end{bmatrix}'$$

$$= \begin{bmatrix} -2 & -2 & 4 \\ -2 & 4 & 4 \\ 1 & -2 & -5 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-6}\begin{bmatrix} -2 & -2 & 4 \\ -2 & 4 & 4 \\ 1 & -2 & -5 \end{bmatrix} \text{...(1)}$$

(II) The given system of equations is :

$$2x + 3y + 4z = 17$$
$$x - y = 3$$
$$y + 2z = 7.$$

These equations can be written as AX = B,

where $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 17 \\ 3 \\ 7 \end{bmatrix}$.

Since A is non-singular, $[\because |A| \neq 0]$

$\therefore$ the given system has a unique solution given by $X = A^{-1}B$

$[\because AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B]$

i.e. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{6}\begin{matrix} -2 & -2 & 4 \\ -2 & 4 & 4 \\ 1 & -2 & -5 \end{matrix}\begin{bmatrix} 17 \\ 3 \\ 7 \end{bmatrix}$ [*Using (1)*]

$$= -\frac{1}{6}\begin{bmatrix} -34-6+28 \\ -34+12+28 \\ 17-6-35 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} -12 \\ 6 \\ -24 \end{bmatrix}$$

$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$. Hence, $x = 2$, $y = -1$ and $z = 4$.

Q. 31. Minimise and Maximise Z = $x + 2y$ subject to $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$, $x, y \geq 0$.

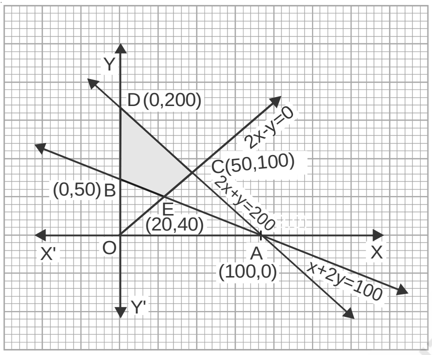
Ans. The system of constraints is :

$x + 2y \geq 100$ ...(1)

$2x - y \leq 0$ ...(2)

$2x + y \leq 200$ ...(3)

and $x, y \geq 0$ ...(4)



The shaded region in the above figure is the feasible region determined by the system of constraints (1)–(4). It is observed that the feasible region ECDB is bounded. Thus we use ***Corner Point Method*** to determine the maximum of Z, where :

$Z = x + 2y$ ...(5)

The co-ordinates of E, C, D and B are

(20, 40)(on solving $x + 2y = 100$ and $2x - y = 0$)

(50, 100)(on solving $2x + y = 200$ and $2x - y = 0$)

(0, 200) and (0, 50) respectively.

Corner point	Corresponding Value of Z
E : (20, 40)	**100 (Minimum)**
C : (50, 100)	250
D : (0, 200)	**400 (Maximum)**
B : (0, 50)	**100 (Minimum)**

Hence, $Z_{max} = 400$ at (0, 200)

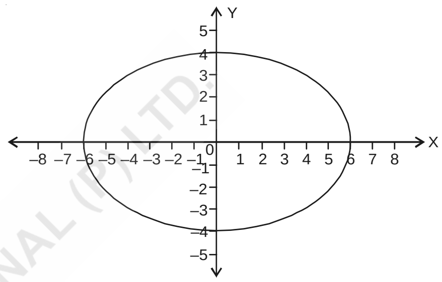
and $Z_{min} = 100$ at all points on the line segment joining the points B (0, 50) and E (20, 40).

SECTION—D

(This section comprises of Long Answer Type questions (LAQ) of 5 marks each)

Q. 32. Shown below the concrete elliptical water pipes, each of 10 feet is length.

Concrete elliptical pipes



Inner circumference of the pipe

The graph given above represents the inner circumference of the elliptical pipe, where x and y are in feet. Assume hat the water flows uniformly and fully covers the inner cross-sectional area of the pipe.

Find the volume of water in the pipe at a given instant of time, in terms of π. Use the integration method and show your steps.

(*Note : Volume = Area of the base × Height)*

Ans. The equation of the ellipse is :

$$\frac{x^2}{6^2} + \frac{y^2}{4^2} = 1 \text{ i.e. } \frac{x^2}{36} + \frac{y^2}{16} = 1 \quad ...(1)$$

From (1), $\frac{y^2}{16} = 1 - \frac{x^2}{36}$

$\Rightarrow$ $y = \pm\sqrt{\frac{16}{36}}\sqrt{36 - x^2}$

or $y = \pm\frac{4}{6}\sqrt{36 - x^2}$.

Area of the quarter of the ellipse

$$= \int_0^6 \frac{4}{6}\sqrt{36 - x^2}\, dx$$

$$= \frac{4}{6}\left[\frac{x\sqrt{36-x^2}}{2}+\frac{36}{2}\sin^{-1}\frac{x}{6}\right]_0^6$$

$$= \frac{2}{3}[\{0+18\sin^{-1}(1)\}-\{0+0\}]$$

$$= \frac{2}{3}\left(18\times\frac{\pi}{2}\right) = 6\pi.$$

(*b*) The area of the whole ellipse = 4 (6π) = 24π sq. feet.

Hence, the volume of water in the pipe.

= Area of the base × Height

= 24π × 10 = 240 cubic feet.

Q. 33. Show that the relation R defined in the set A of all polygons as R = {(P_1, P_2) : P_1 and P_2 have same number of sides}, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5 ?

Ans. We have : R = {(P_1, P_2), P_1 and P_2 have same number of sides}.

R is reflexive. [∵ *P and P have same number of sides ⇒ (P, P) ∈ R ∀ P ∈ A*]

R is symmetric. [∵ *(P_1, P_2) ∈ R ⇒ P_1 and P_2 have same number of sides*

⇒ P_2 and P_1 have same number of sides

⇒ (P_2, P_1) ∈ R]

R is transitive. [∵ *(P_1, P_2) ∈ R and (P_2, P_3) ∈ R ⇒ P_1 and P_2 have same number of sides & P_2 and P_3 have same number of sides ⇒ P_1 and P_3 have same number of sides ⇒ (P_1, P_3) ∈ R*]

Hence, R is an equivalence relation.

Here T is a triangle.

∴ P ∈ A is related to T iff P and T have same number of sides

⇒ (P, T) ∈ R iff P and T have same number of sides

⇒ (P, T) ∈ R iff P is a triangle.

Hence, the set of all elements, which are related to T is the set of all triangles in A.

Or

Let A = R – {3} and B = R – {1}. Consider the function : *f* : A → B defined by *f*(*x*) = $\left(\frac{x-2}{x-3}\right)$. Is *f* one-one and onto ? Justify your answer.

Ans. Let $x_1, x_2 \in \mathbf{R} - \{3\}$.

Now $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$$

$$\Rightarrow (x_1-2)(x_2-3) = (x_2-2)(x_1-3)$$

$$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_2x_1 - 3x_2 - 2x_1 + 6$$

$$\Rightarrow -3x_1 - 2x_2 = -3x_2 - 2x_1$$

$$\Rightarrow x_1 = x_2$$

⇒ *f* is one-one.

Let $y \in \mathbf{R} - \{1\}$.

Then $f(x) = y$.

When $\frac{x-2}{x-3} = y,\ x \neq 3$

$$\Rightarrow x - 2 = yx - 3y$$

$$\Rightarrow x - xy = 2 - 3y$$

$$\Rightarrow x = \frac{2-3y}{1-y} \in A$$

$$\left[\because \frac{2-3y}{1-y} = 3 - \frac{1}{1-y} \neq 3\right]$$

∴ Corresponding to each $y \in$ B, there exists $\frac{2-3y}{1-y} \in$ A such that $f\left(\frac{2-3y}{1-y}\right) = y$

⇒ *f* is onto.

Hence, '*f*' is one-one and onto.

Q. 34. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.

Ans. Let the position vectors of vertices A, B and C be $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively.

$$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (\hat{i}-3\hat{j}-5\hat{k}) - (2\hat{i}-\hat{j}+\hat{k})$$

$$= -\hat{i}-2\hat{j}-6\hat{k}.$$

$$\therefore AB = |\overrightarrow{AB}|$$

$$= \sqrt{(-1)^2+(-2)^2+(-6)^2}$$

$$= \sqrt{1+4+36} = \sqrt{41},$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= (3\hat{i}-4\hat{j}-4\hat{k}) - (\hat{i}-3\hat{j}-5\hat{k})$$

$$= 2\hat{i}-\hat{j}+\hat{k}\cdot$$

$$\therefore BC = |\overrightarrow{BC}| = \sqrt{2^2+(-1)^2+1^2}$$

$$= \sqrt{4+1+1} = \sqrt{6}$$

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$$= (3\hat{i}-4\hat{j}-4\hat{k}) - (2\hat{i}-\hat{j}+\hat{k})$$

$$= \hat{i}-3\hat{j}-5\hat{k}\cdot$$

$$\therefore AC = |\overrightarrow{AC}| = \sqrt{1^2+(-3)^2+(-5)^2}$$

$$= \sqrt{1+9+25} = \sqrt{35}\cdot$$

Now $BC^2 + AC^2$ = 6 + 35 = 41 = AB.

Hence, ΔABC is a right angled triangle.

Or

Find the vector and the cartesian equations of the line that passes through the point (3, –2, –5), (3, –2, 6).

Ans. Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A (3, – 2, – 5) and B (3, – 2, 6).

Then $\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$

and $\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$.

$\therefore \quad \vec{b} - \vec{a} = 11\hat{k}$.

Let $\vec{r}$ be the position vector of any point on the line.

Then the vector equation of the line is :

$$\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$$

i.e. $\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda\left(11\hat{k}\right)$

$\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = 3\hat{i} - 2\hat{j} + (11\lambda - 5)\hat{k}$.

Comparing, $x = 3, y = -2$ and $z = 11\lambda - 5$.

Eliminating λ, $\dfrac{x-3}{0} = \dfrac{y+2}{0} = \dfrac{z+5}{11}$ $(= \lambda)$,

which is the cartesian from of the equation of the line.

Q. 35. A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Determine

(*i*) *k* **(*ii*) P(X < 3)**

(*iii*) P(X > 6) **(*iv*) P(0 < X < 3).**

Ans. (*i*) Since $\Sigma P(X) = 1$,

$\therefore 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow \quad 10k^2 + 9k - 1 = 0$

$\Rightarrow \quad k = \dfrac{-9 \pm \sqrt{81+40}}{20}$

$= \dfrac{-9 \pm 11}{20} = \dfrac{1}{10}, -1$.

Since the probability is ≥ 0, therefore, rejecting K = – 1, we have : K = $\dfrac{1}{10}$.

(*ii*) (I) P (X < 3) = P (X = 0) + P (X = 1) + P (X = 2)

$= 0 + k + 2k = 3k = \dfrac{3}{10}$.

(II) P (X > 6) = P (7)

$= 7k^2 + k$

$= \dfrac{7}{100} + \dfrac{1}{10} = \dfrac{17}{100}$.

(III) P (0 < X < 3) = P (X = 1) + P (X = 2)

$= k + 2k = 3k = \dfrac{3}{10}$.

SECTION—E

(This section comprises of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36.Read the following passage and answer the questions, which follow :

An organization conducted bike race under 2 different categories-boys and girls. Totally there were 250 participants. Among all of them finally three from Category 1 and two from Category 2 were selected for the final race. Ravi forms two sets B and G with these participants for his college project.

Let B = $\{b_1, b_2, b_3\}$ G = $\{g_1, g_2\}$ where B represents the set of boys selected and G the set of girls who were selected for the final race.

Ravi decides to explore these sets for various types of relations and functions.

Based on the above information, answer the following :

(I) Ravi wishes to form all the relations possible from B to G. How many such relations are possible ?

(II) Let R : B → B be defined by R = {(*x* ,*y*): *x* and *y* are students of same sex}. What is this relation R is ..... .

(III) Ravi wants to know among those relations, how many functions can be formed from B to G ?

Or

Ravi wants to find the number of injective functions from B to G. How many numbers of injective functions are possible ?

Ans. (I) 2^6 **(II)** equivalence **(III)** 2^3 **Or** 0.

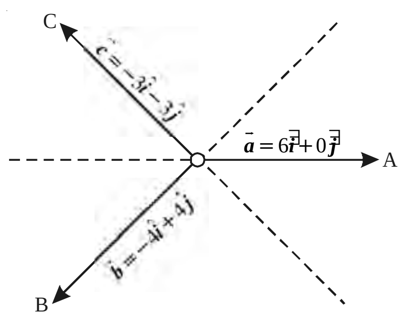
Q. 37. Read the following passage and answer the questions given below :

Teams A, B, C went for playing a tug of war game. Teams A, B, C have attached a rope to a metal ring and is trying to pull the ring into their own area.

Team A pulls with force $F_1 = 6\hat{i} + 0\hat{j}\ kN$,

Team B pulls with force $F_2 = -4\hat{i} + 4\hat{j}\ kN$,

Team C pulls with force $F_3 = -3\hat{i} - 3\hat{j}\ kN$,



(I) What is the magnitude of the force of Team A ?

(II) Which team will win the game?

(III) Find the magnitude of the resultant force exerted by the terms.

Or

In what direction is the ring getting pulled?

Ans. Here, $|\vec{F}_1| = \sqrt{6^2+0^2} = 6\ kN$

$$|\vec{F}_2| = \sqrt{(-4)^2+4^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}\ kN$$

$$|\vec{F}_3| = \sqrt{(-3)^2+(-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}\ kN$$

(I) Magnitude of force of Team A = 6 kN.

(II) Since, 6 kN is greatest,

$\therefore$ team (A) will win the game.

(III) $|\vec{F}_1| = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$

$$= 6\hat{i} + \hat{j} - 4\hat{i} + 4\hat{j} - 3\hat{i} - 3\hat{j} = -\hat{i} + \hat{j}$$

$$\therefore\ |\vec{F}| = \sqrt{(-1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}\ kN.$$

Or

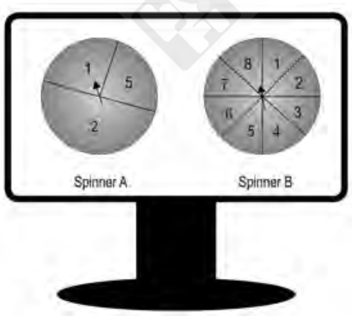
Here, $\vec{F} = -\hat{i} + \hat{j}$

$$\therefore\ \theta = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4},$$

Where 'θ' is the angle made by the resultant force with the +ve direction of x-axis.

Q. 38. Answer the questions based on the given information.

Rubiya. Thaksh, Shanteri, and Lilly entered a spinning zone for a fun game, but there is a twist: they don't know which spinner will appear on their screens until it is their turn to play. They may encounter one of the following spinners, or perhaps ever both :



Different combinations of numbers will lead to exciting prizes. Below are some of the rewards they can win:

• Get the number '5', from Spinner A and '8' from Spinner B, and you'll win a music player!

• You win a photo frame if Spinner A lands on a value greater than that of Spinner B!

(I) Thaksh spun both the spinners, A and B in one of his turns.

What is the probability that Thaksh wins a music player in that turn ? Show your steps.

(II) Lilly spun spinner B in one of her turns.

What is the probability that the number she got is even given that it is a multiple of 3 ? Show your steps.

(III) Rubiya spun both the spinners.

What is the probability that she wins a photo frame ? Show your work.

Or

As Shanteri steps up to the screen, the game administrator reveals that for her turn, the probability of seeing Spinner A on the screen is 65%, while that of Spinner B is 35%.

What is the probability that Shanteri gets the number '2' ? Show your steps.

Ans. (I) P (5 from spinner A) $\cap$ P (8 from spinner B)

$$= \frac{1}{4} \times \frac{1}{8} = \frac{1}{32}.$$

(II) P (Even/Multiple of 3)

$$= \frac{\text{P (Even and Multiple of 3)}}{\text{P (Multiple of 3)}}$$

$$= \frac{1/8}{2/8} = \frac{1}{2}.$$

(III) Here P_1 = (getting 2 from spinner A and 1 from spinner B)

$$= \frac{1}{2} \times \frac{1}{8} = \frac{1}{16}.$$

P_2 = (getting 5 from spinner A) and (either 1, 2, 3 or 3 from spinner B)

$$= \frac{1}{4} \times \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= \frac{1}{4} \times \frac{4}{8} = \frac{1}{8}.$$

Since P_1 and P_2 are mutually exclusive,

$\therefore$ P (wins a photo fame) = $P_1 + P_2$

$$= \frac{1}{16} + \frac{1}{8} = \frac{3}{16}.$$

Or

P (Getting 2) = [P (Spinner 2) × P (Getting 2/Spinner A)] + [P (Spinner B) × P (Getting 2/Spinner B)]

$$= \left[\frac{65}{100} \times \frac{1}{2}\right] + \left[\frac{35}{100} + \frac{1}{2}\right]$$

$$= \frac{59}{160}.$$

Holy Faith New Style Sample Paper–9

(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions : Same as Holy Faith New Style Sample Paper–1.

SECTION—A

(Multiple choice questions, each question carries 1 mark)

1. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Based on the given information, f is best defined as :

(*a*) Surjective function (*b*) Injective function
(*c*) Bijective function (*d*) Function.

Ans. (*b*) Injective function.

2. If $\sin^{-1} x = y$, then :

(*a*) $0 \le y \le \pi$ (*b*) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(*c*) $0 < y < \pi$ (*d*) $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Ans. (*b*) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

3. Given that matrices A and B are of order $3 \times n$ and $m \times 5$ respectively, then the order of the matrix : C = 5A + 3B is :

(*a*) 3×5 and $m = n$ (*b*) 3×5
(*c*) 3×3 (*d*) 5×5.

Ans. (*b*) 3×5.

4. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to :

(*a*) A (*b*) I + A
(*c*) I – A (*d*) I.

Ans. (*c*) I – A.

5. Given that A is a square matrix of order 3 and | A | = – 4, then | *adj* A | is equal to :

(*a*) – 4 (*b*) 4
(*c*) –16 (*d*) 16.

Ans. (*d*) 16.

6. For what value of '*k*' may the function

$$f(x) = \begin{cases} k(3x^2 - 5x), & x \le 0 \\ \cos x, & x > 0 \end{cases}$$ become continuous ?

(*a*) 0 (*b*) 1

(*c*) $-\frac{1}{2}$ (*d*) No value.

Ans. (*d*) No value.

7. If $f(x) = \cos^{-1}\sqrt{x}$, $0 < x < 1$, which of the following is equal to $f'(x)$?

(*a*) $\frac{-1}{\sqrt{1-x}}$ (*b*) $\frac{1}{\sqrt{1-x}}$

(*c*) $\frac{1}{2\sqrt{x(1-x)}}$ (*d*) $\frac{-1}{2\sqrt{x(1-x)}}$.

Ans. (*d*) $\frac{-1}{2\sqrt{x(1-x)}}$.

8. The area of a trapezium is defined by function f and given by :

$$f(x) = (10 + x)\sqrt{100 - x^2}.$$

Then the area when it is maximised is :

(*a*) 75 cm^2 (*b*) $7\sqrt{3}$ cm^2

(*c*) $75\sqrt{3}$ cm^2 (*d*) 5 cm^2.

Ans. (*c*) $75\sqrt{3}$ cm^2.

9. $\int \sqrt{x^2 - 8x + 7}\, dx$ is equal to :

(*a*) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} + 9\log\left|(x-4) + \sqrt{x^2-8x+7}\right| + c$

(*b*) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7} + 9\log\left|(x+4) + \sqrt{x^2-8x+7}\right| + c$

(*c*) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - 3\sqrt{2}\log\left|x-4 + \sqrt{x^2-8x+7}\right| + c$

(*d*) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - \frac{9}{2}\log\left|x-4 + \sqrt{x^2-8x+7}\right| + c$.

Ans. (*d*) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}$

$$-\frac{9}{2}\log|x-4+\sqrt{x^2-8x+7}|+c.$$

10. $\int_{-\pi/2}^{\pi/2}\frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$, where $x \neq 0$, is equal to :

(*a*) – 2 (*b*) 0
(*c*) 1 (*d*) π.

Ans. (*b*) 0.

11. The area bounded by the curve $y = 4\sin x$, x-axis from $x = 0$ to $x = \pi$ is equal to :

(*a*) 1 sq. units (*b*) 2 sq. units
(*c*) 4 sq. units (*d*) 8 sq. units.

Ans. (*d*) 8 sq. units.

12. The integrating factor of the differential equation $x\frac{dy}{dx}-y=2x^2$ is :

(*a*) e^{-x} (*b*) e^{-y}
(*c*) $\frac{1}{x}$ (*d*) x.

Ans. (*c*) $\frac{1}{x}$.

13. The solution of the differential equation : $x\frac{dy}{dx}+2y = x^2$ is :

(*a*) $y=\frac{x^2+c}{4x^2}$ (*b*) $y=\frac{x^2}{4}+c$
(*c*) $y=\frac{x^4+c}{x^2}$ (*d*) $y=\frac{x^4+c}{4x^2}$.

Ans. (*d*) $y=\frac{x^4+c}{4x^2}$.

14. The value of $(\hat{i}\times\hat{j}).\hat{j}+(\hat{j}\times\hat{i}).\hat{k}$ is :

(*a*) 2 (*b*) 0
(*c*) 1 (*d*) –1.

Ans. (*d*) –1.

15. If $|\vec{a}|=4$ and $-3 \le \lambda \le 2$, then $|\lambda\vec{a}|$ lies in :

(*a*) [0, 12] (*b*) [2, 3]
(*c*) [8,12] (*d*) [–12, 8].

Ans. (*d*) [–12, 8].

16. If a line makes angles of 90°, 135° and 45° with the x, y and z-axes respectively, then its direction-cosines are :

(*a*) $<0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}>$ (*b*) $<-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}>$
(*c*) $<\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}>$ (*d*) $<0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}>$.

Ans. (*a*) $<0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}>$

17. A Linear Programming Problem is as follows :
Maximise/Minimise objective function $Z = 2x - y + 5$.
subject to the constraints :

$3x + 4y \le 60$
$x + 3y \le 30$
$x \ge 0, y \ge 0.$

If the corner points of the feasible region are A (0, 10), B (12, 6), C (20, 0) and O (0, 0), then which of the following is true ?

(*a*) Maximum value of Z is 40
(*b*) Minimum value of Z is – 5
(*c*) Difference of maximum and minimum values of Z is 35
(*d*) At two corner points, value of Z are equal.

Ans. (*b*) Minimum value of Z is – 5

18. From the set {1, 2, 3, 4, 5}, two numbers a and b ($a \neq b$) are chosen at random. The probability that $\frac{a}{b}$ is an integer is :

(*a*) $\frac{1}{3}$ (*b*) $\frac{1}{4}$
(*c*) $\frac{1}{2}$ (*d*) $\frac{3}{5}$.

Ans. (*c*) $\frac{1}{2}$.

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) Both ‘A’ and ‘R’ are true and ‘R’ is the correct explanation of ‘A’.
(*b*) Both ‘A’ and ‘R’ are true but ‘R’ is not the correct explanation of ‘A’.
(*c*) ‘A’ is true but ‘R’ is false.
(*d*) ‘A’ is false but ‘R’ is true.

19. Assertion (A) : $\sin^{-1}2x+\sin^{-1}3x=\frac{\pi}{3}=x=\sqrt{\frac{3}{76}}$.

Reason (R) : Sum of two negative angles can’t be equal to positive.

Ans. (*c*) $\sin^{-1}2x+\sin^{-1}3x=\frac{\pi}{3}$

$$\Rightarrow \frac{\pi}{2}-\cos^{-1}2x+\frac{\pi}{2}-\cos^{-1}3x=\frac{\pi}{3}$$

$$\Rightarrow \cos^{-1}2x+\cos^{-1}3x=\frac{2\pi}{3}$$

$$\Rightarrow \cos^{-1}(6x^2-\sqrt{1-4x^2}\sqrt{1-9x^2})=\frac{2\pi}{3}$$

$$\Rightarrow 6x^2-\sqrt{1-13x^2+36x^4}=-\frac{1}{2}$$

$$\Rightarrow \left(6x^2+\frac{1}{2}\right)^2=1-13x^2+36x^4$$

$$\Rightarrow \quad 36x^4 + 6x^2 + \frac{1}{4} = 1 - 13x^2 + 36x^4$$

$$\Rightarrow \quad 19x^2 = \frac{3}{4}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{3}{76}}$$

But sum of triangular numbers $\neq \frac{\pi}{3}$.

Hence, $x = \sqrt{\frac{3}{76}}$.

But (R) is false.

20. **Assertion (A) :** The points A (2, 9, 12), B (1, 8, 8), C (– 2, 11, 8) and D (– 1, 12, 12) are the vertices of a rhombus.

Reason (R) : AB = BC = CD = DA and AC $\neq$ BD.

Ans. (*d*) $AB = \sqrt{(2-1)^2+(9-8)^2+(12-8)^2} = \sqrt{18} = 3\sqrt{2}$.

Similarly $BC = 3\sqrt{2}$, $CD = 3\sqrt{2}$ and $DA = 3\sqrt{2}$.

Also, AC = 6 and BD = 6.

Thus, AB = BC = CD = DA and AC = BD.

$\therefore$ ABCD is not a rhomlus.

Thus (A) is false.

But (R) is true.

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Solve : $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$.

Ans. The given equation is :

$$\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2} \qquad ...(1)$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{12}{x}\right)$$

$$\Rightarrow \quad \frac{12}{x} = \sin\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{5}{x}\right)\right]$$

$$\Rightarrow \quad \frac{12}{x} = \cos\left(\sin^{-1}\left(\frac{5}{x}\right)\right) = \sqrt{1-\left(\frac{5}{x}\right)^2}$$

$$\left[\because \cos(\sin^{-1}x) = \sqrt{1-x^2}, |x| < 1\right]$$

$$\Rightarrow \left(\frac{12}{x}\right)^2 = 1 - \left(\frac{5}{x}\right)^2 \Rightarrow 144 = x^2 - 25 \Rightarrow x^2 = 169.$$

Hence, $x = 13$. [$\because$ *$x = -13$ does not satisfy (1)*]

Or

Show that : $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$.

Ans. Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$.

$$\therefore \quad \sin x = \frac{3}{5} \quad \text{and} \quad \sin y = \frac{8}{17}$$

so that $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

and $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$.

Now $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$$= \left(\frac{4}{5}\right)\left(\frac{15}{17}\right) + \left(\frac{3}{5}\right)\left(\frac{8}{17}\right) = \frac{60+24}{85} = \frac{84}{85}$$

$$\Rightarrow \quad x - y = \cos^{-1}\left(\frac{84}{85}\right).$$

Hence, $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$.

Q. 22. The length *x* of a rectangle is decreasing at the rate of 3 cm/minute and the width *y* is increasing at the rate of 2 cm/minute. When *x* = 10 cm and *y* = 6 cm, find the rates of change of (*a*) the perimeter, and (*b*) the area of the rectangle.

Ans. We have : $\frac{dx}{dt} = -3 \qquad ...(1)$

and $\frac{dy}{dt} = 2 \qquad ...(2)$

(*a*) Perimeter, $p = 2x + 2y$.

$$\therefore \quad \frac{dp}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt}$$

$= 2(-3) + 2(2)$ [*Using (1) and (2)*]

$= -6 + 4 = -2.$

Hence, $\left.\frac{dp}{dt}\right]_{\substack{x=10\\y=6}} = -2$ cm/m.

(*b*) Area, $A = xy$.

$$\therefore \quad \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$= x(2) + y(-3)$. [*Using (1) and (2)*]

Hence, $\left.\frac{dA}{dt}\right]_{\substack{x=10\\y=6}} = 10(2) + 6(-3) = 20 - 18 = 2\ \text{cm}^2/\text{m}.$

Q. 23. Prove that the function $g(x) = \log x$ do not have maxima or minima.

Ans. We have $f(x) = \log x$.

$$\therefore \quad f'(x) = \frac{1}{x}.$$

Now $f'(x) = 0 \Rightarrow \frac{1}{x} = 0$,

which gives no real value of x.

Hence, $f(x)$ does not have maximum or minimum values.

Or

Determine the absolute maximum and absolute minimum values of the following in the stated domains :

$$y = \frac{1}{2}x^2 + 5x + \frac{3}{2}; -6 \leq x \leq -2$$

Ans. We have : $y = \frac{1}{2}x^2 + 5x + \frac{3}{2}$ ...(1)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(2x) + 5(1) + 0$$

$$\Rightarrow \quad \frac{dy}{dx} = x + 5 \quad ...(2)$$

Now $\frac{dy}{dx} = 0$ gives : $x + 5 = 0 \Rightarrow x = -5 \in [-6, -2]$.

Now $$y\Big]_{x=-6} = \frac{1}{2}(-6)^2 + 5(-6) + \frac{3}{2}$$

$$= 18 - 30 + \frac{3}{2} = -\frac{21}{2}$$

$$y\Big]_{x=-5} = \frac{1}{2}(-5)^2 + 5(-5) + \frac{3}{2}$$

$$= \frac{25}{2} - 25 + \frac{3}{2} = -11$$

and $$y\Big]_{x=-2} = \frac{1}{2}(-2)^2 + 5(-2) + \frac{3}{2}$$

$$= 2 - 10 + \frac{3}{2} = -\frac{13}{2}.$$

Hence, the absolute max. value of $y = -\frac{13}{2}$ and absolute min. value of $y = -11$.

Q. 24. If $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$, find the value of '$a$'.

Ans. $$\int_0^a \frac{1}{4+x^2}dx = \int_0^a \frac{dx}{2^2+x^2}$$

"Form : $\int \frac{dx}{a^2+x^2}$"

$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^a$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{a}{2} - \tan^{-1}(0)\right]$$

$$= \frac{1}{2}\tan^{-1}\frac{a}{2}.$$

By the question,

$$\frac{1}{2}\tan^{-1}\frac{a}{2} = \frac{\pi}{8} \quad \Rightarrow \quad \tan^{-1}\frac{a}{2} = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{a}{2} = \tan\frac{\pi}{4} \quad \Rightarrow \quad \frac{a}{2} = 1.$$

Hence, $a = 2$.

Q. 25. Differentiate $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ w.r.t. x.

Ans. Let $$y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$$

$$= \sin^{-1}\left(\frac{2.2^x}{1+(2^x)^2}\right).$$

Put $\mathbf{2^x = \tan\theta}$

so that $\theta = \tan^{-1}(2^x)$ ...(1)

Then, $$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1}(2^x). \text{ [Using (1)]}$$

$$\therefore \quad \frac{dy}{dx} = 2\frac{1}{1+(2^x)^2}.\frac{d}{dx}(2^x)$$

$$= \frac{2}{1+4^x}.2^x.\log 2$$

$$= \frac{2^{x+1}\log 2}{1+4^x}.$$

SECTION—C

(This section comprises of short answer type questions (SAQ) of 3 marks each)

Q. 26. Evaluate : $\int(\sqrt{\tan x} + \sqrt{\cot x})\,dx$.

Ans. $$I = \int(\sqrt{\tan x} + \sqrt{\cot x})\,dx$$

$$= \int \frac{\tan x + 1}{\sqrt{\tan x}}dx.$$

Put $\sqrt{\tan x} = t$ *i.e.*, $\tan x = t^2$

so that $\sec^2 x\,dx = 2t\,dt$

i.e. $(1 + \tan^2 x)\,dx = 2t\,dt$ *i.e.* $(1 + t^4)\,dx = 2t\,dt$

i.e. $$dx = \frac{2t}{1+t^4}dt.$$

$$\therefore \quad I = \int \frac{t^2+1}{t(1+t^4)}2t\,dt$$

$$= 2\int \frac{t^2+1}{t^4+1}dt = 2\int \frac{1+1/t^2}{t^2+\frac{1}{t^2}}dt.$$

[Dividing Num.. & Denom. by t^2]

Put $t - \frac{1}{t} = y$ so that $\left(1+\frac{1}{t^2}\right)dt = dy$.

Also, $t^2 - 2 + \frac{1}{t^2} = y^2$

$\Rightarrow \quad t^2 + \frac{1}{t^2} = y^2 + 2.$

$\therefore \quad I = 2\int \frac{dy}{(\sqrt{2})^2 + y^2}$

$\left[\text{"Form : } \int \frac{dx}{a^2 + x^2}\text{"}\right.$

$= \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + c$

$= \sqrt{2}\tan^{-1}\left(\frac{t - 1/t}{\sqrt{2}}\right) + c$

$= \sqrt{2}\tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right) + c$

$= \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + c.$

Q. 27. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Ans. Let n be the number of throws when a multiple of 3 occurs.

$\therefore$ P (a multiple of 3 *i.e.* 3 or 6) in one throw $= \frac{2}{6} = \frac{1}{3}$

P (a multiple of 3) in n throws $= \left(\frac{1}{3}\right)^n$.

P (getting a 6) in one throw $= \frac{1}{6}$

P (getting a 6) in n throws $= \left(\frac{1}{6}\right)^n$.

$\therefore$ Probability of getting at least a 3 in n throws

$= \left(\frac{1}{3}\right)^n - \left(\frac{1}{6}\right)^n.$

Let a multiple of 3 do not occur in $(n + 1)$th throw.
P (getting 1, 2, 4, 5) in $(n + 1)$th throw

$= \frac{4}{6} = \frac{2}{3}.$

In the next throw a coin is tossed and tail occurs.

$\therefore$ P (getting a tail) $= \frac{1}{2}$

P (getting at least a 3 and a tail) in $(n + 2)$th throw

$= \left[\left(\frac{1}{3}\right)^n - \left(\frac{1}{6}\right)^n\right]\frac{2}{3} \times \frac{1}{2}$

Since $n \to \infty$, the probability of getting at least a 3 till tail is obtained

$= \sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n - \left(\frac{1}{6}\right)^n\right] \times \frac{2}{3} \times \frac{1}{2}$

$= \left[\left(\frac{\frac{1}{3}}{1 - \frac{1}{3}}\right) - \left(\frac{\frac{1}{6}}{1 - \frac{1}{6}}\right)\right] \times \frac{1}{3} = \left(\frac{1}{3} \times \frac{3}{2} - \frac{1}{6} \times \frac{6}{5}\right) \times \frac{1}{3}$

$= \left(\frac{1}{2} - \frac{1}{5}\right) \times \frac{1}{3} = \frac{3}{10} \times \frac{1}{3} = \frac{1}{10}.$

Q. 28. Find : $\int \frac{3x + 5}{x^2 + 4x + 7}dx$.

Ans. Let $3x + 5 = A(2x + 4) + B$.

Comparing, $2A = 3, 4A + B = 5$

$\Rightarrow \quad A = \frac{3}{2}, B = 5 - 4\left(\frac{3}{2}\right)$

$= 5 - 6 = -1.$

$\therefore \int \frac{3x + 5}{x^2 + 4x + 17}dx = \int \frac{\frac{3}{2}(2x + 4) - 1}{x^2 + 4x + 7}dx$

$\therefore I = \frac{3}{2}\int \frac{2x + 4}{x^2 + 4x + 7}dx - \int \frac{1}{x^2 + 4x + 7}dx \quad ...(1)$

Now $I_1 = \int \frac{2x + 4}{x^2 + 4x + 7}dx.$

Put $x^2 + 4x + 7 = t$ so that $(2x + 4)\,dx = dt$.

$\therefore \quad I_1 = \int \frac{dt}{t} = \log|t| = \log|x^2 + 4x + 7|.$

And, $I_2 = \int \frac{1}{x^2 + 4x + 7}dx$

$= \int \frac{1}{(x^2 + 4x + 4) + (7 - 4)}$

$= \int \frac{dx}{(x + 2)^2 + (\sqrt{3})^2}.$

Put $x + 2 = y$ so that $dx = dy$.

$\therefore \quad I_2 = \int \frac{dy}{\sqrt{3} + y^2}$ | **"Form :** $\int \frac{dx}{a^2 + x^2}$"

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{y}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{(x+2)}{\sqrt{3}}.$$

Putting in (1), $I = \frac{3}{2} \log |x^2 + 4x + 17|$

$$- \frac{1}{\sqrt{3}} \tan^{-1} \frac{(x+2)}{\sqrt{3}} + c.$$

Or

Evaluate : (*i*) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

(*ii*) Evaluate : $\int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x} dx$ **.**

Ans. (*i*) Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ ...(1)

$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1+\cos^2 (\pi - x)} dx$ [*Property V*]

$$= \int_0^{\pi} \frac{(\pi - x) \sin x}{1+\cos^2 x} dx$$

$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - I$ [*Using (1)*]

$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$ ...(2)

Now $I_1 = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$.

Put cos x = t so that $-\sin x\, dx = dt$

i.e. $\sin x\, dx = -dt$.

When $x = 0$, $t = \cos 0 = 1$.

When $x = \pi$, $t = \cos \pi = -1$.

$\therefore \; I_1 = \int_1^{-1} \frac{-dt}{1+t^2} = \int_{-1}^{1} \frac{dt}{1+t^2}$ [*Property II*]

$$= \left[\tan^{-1} t\right]_{-1}^{1} = \tan^{-1} 1 - \tan^{-1} (-1)$$

$$= \tan^{-1} 1 + \tan^{-1} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

$\therefore$ From (2), $2I = \pi \left(\frac{\pi}{2}\right) \Rightarrow 2I = \frac{\pi^2}{2}$.

Hence, $I = \frac{\pi^2}{4}$.

(*ii*) $I = \int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x} dx$

$$= \int_0^{\pi} \frac{4(\pi - x) \sin (\pi - x)}{1+\cos^2 x} dx$$

$$= \int_0^{\pi} \frac{4(\pi - x) \sin x}{1+\cos^2 x} dx$$

$$= 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - I$$

$\Rightarrow \quad 2I = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

$\Rightarrow \quad I = 2\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$ ...(1)

$$I_1 = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dy$$

Put $\cos x = t$ so that $-\sin x\, dx = dt$

i.e., $\sin dx = -dt$

when $x = 0$, $t = 1$, when $x = \pi$, $t = -1$

$\therefore \quad I_1 = \int_1^{-1} \frac{-dt}{1+\tan^2} dt$

$$= -[\tan^{-1} t]^{-1}$$

$$= -[\tan^{-1} (-1) - \tan^{-1} (1)]$$

$$= -\left[\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi}{2}$$

From (1), $I = 2\pi\left(\frac{\pi}{2}\right) = \pi^2$.

Q. 29. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after *t* seconds.

Ans. Let '*r*' be the radius of spherical balloon at any time '*t*'.

$\therefore \quad V = \frac{4}{3} \pi r^3$.

By the question, $\frac{dV}{dt} = k$ (constant)

$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3} \pi r^3\right) = k$

$\Rightarrow \quad \frac{4}{3} \pi (3r^2) \frac{dr}{dt} = k \quad \Rightarrow \quad 4\pi r^2\, dr = k\, dt$.

|*Variables Separable*

Integrating, $4\pi\int r^2 dr = k\int 1.dt + c$

$$\Rightarrow \quad 4\pi\frac{r^3}{3} = kt + c \quad ...(1)$$

When $t = 0$, $r = 3$.

$$\therefore \quad \frac{4\pi}{3}(27) = 0 + c \quad \Rightarrow \quad c = 36\pi.$$

When $t = 3$, $r = 6$.

$$\therefore \quad \frac{4\pi}{3}(216) = 3k + c \quad \Rightarrow \quad 288\pi = 3k + 36\pi$$

$$\Rightarrow \quad 3k = 252\pi \quad \Rightarrow \quad k = 84\pi.$$

Putting in (1), $\frac{4\pi}{3}r^3 = 84\pi t + 36\pi \Rightarrow \frac{4}{3}r^3 = 84t + 36$

$\Rightarrow r = [3(21t + 9)]^{1/3}$

$\Rightarrow r = [9(7t + 3)]^{1/3}$, which is the radius after 't' seconds.

Or

Show that the given differential equations is $\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}ydx = \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}xdy.$

Ans. The given equation is :

$$\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}y\,dx = \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}x\,dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}}{x\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}} \quad ...(1)$$

Now $f(x, y) = \dfrac{y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}}{x\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}}$.

$$\therefore f(\lambda x, \lambda y) = \frac{\lambda y\left\{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right) + \lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}}{\lambda x\left\{\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right) - \lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)\right\}}$$

$$= \frac{\lambda^2 y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}}{\lambda^2 x\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}}$$

$$= \lambda^0\frac{y\left[x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right]}{x\left[y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right]} = \lambda^0 f(x, y).$$

$\therefore f(x, y)$ is homogeneous function of degree zero.

To solve :

Put y = vx so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ (1) becomes :

$$v + x\frac{dv}{dx} = \frac{vx\left[x\cos\left(\frac{vx}{x}\right) + vx\sin\left(\frac{vx}{x}\right)\right]}{x\left[vx\sin\left(\frac{vx}{x}\right) - x\cos\left(\frac{vx}{x}\right)\right]}$$

$$= \frac{vx^2[\cos v + v\sin v]}{x^2(v\sin v - \cos v)}$$

$$= \frac{v(\cos v + v\sin v)}{v\sin v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{v\sin v - \cos v} - v$$

$$= \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$= \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad \frac{v\sin v - \cos v}{v\cos v}dv = 2\frac{dx}{x}. \quad \textit{|Variables Separable}$$

Integrating, $\int\left(\tan v - \frac{1}{v}\right)dv = 2\int\frac{dx}{x} + \log|C'|$

$\Rightarrow \log|\sec v| - \log|v| = 2\log|x| + \log|C'|$

$$\Rightarrow \quad \log\left|\frac{\sec v}{v}\right| = \log|C'x^2|$$

$$\Rightarrow \quad \frac{\sec v}{v} = C'x^2$$

$$\Rightarrow \quad \frac{x}{y}\sec\frac{y}{x} = C'x^2$$

$$\Rightarrow \quad \sec\frac{y}{x} = C'xy$$

$\Rightarrow xy\cos\frac{y}{x} = c$, where $C' = \frac{1}{c}$,

which is the reqd. solution.

Q. 30. Find the general solution of the following differential equation :

$x\,dy - (y + 2x^2)\,dx = 0$.

Ans. The given differential equation can be written as :

$$\frac{dy}{dx} = \frac{y + 2x^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 2x \quad ...(1)$$

| Linear Equation

Comparing with $\frac{dy}{dx} + Py = Q$, we get :

$$P = -\frac{1}{x} \quad \text{and} \quad Q = 2x.$$

$\therefore$ $\quad$ I.F. $= e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$.

$\therefore$ The solution is : $y \times \frac{1}{x} = \int (2x)\left(\frac{1}{x}\right)dx$

$\Rightarrow \quad \frac{y}{x} = 2x + c$

$\Rightarrow \quad y = 2x^2 + cx,$

which is the reqd. solution.

Or

If A = $\begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Hence, solve the system of equations :

$$x - 2y = 10$$
$$2x - y - z = 8$$
$$-2y + z = 7.$$

Ans. Here, $\quad A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$.

$\therefore \quad |A| = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix}$

$= 1(-1-2) - 2(-2+0)$
$= -3 + 4 = 1 \neq 0.$

$\therefore$ A is non-singular $\Rightarrow$ A^{-1} exists.

Here $\quad A_{11} = -3,\ A_{12} = 2,\ A_{13} = 2$;
$A_{21} = -2,\ A_{22} = 1,\ A_{23} = 1$;
$A_{31} = -4,\ A_{32} = 2,\ A_{33} = 3$.

$\therefore$ adj. A $= \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$.

$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{Adj}\,A)$

$= \frac{1}{1}\begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ ...(1)

The given system of equations can be written as :

$A'X = B,$

where $A' = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$.

$\therefore \quad X = (A')^{-1}B = (A^{-1})'B$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

$= \begin{bmatrix} -30+16+14 \\ -20+8+7 \\ -40+16+21 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$.

Comparing, $x = 0$, $y = -5$ and $z = -3$.

Q. 31. An aeroplane is flying along the line $\vec{r} = \lambda(\hat{i} - \hat{j} + \hat{k})$; where '$\lambda$' is a scalar and another aeroplane is flying along the line $\vec{r} = \hat{i} - \hat{j} + \mu(-2\hat{j} + \hat{k})$; where '$\mu$' is a scalar. At what points on the lines should they reach, so that the distance between them is the shortest? Find the shortest distance between them.

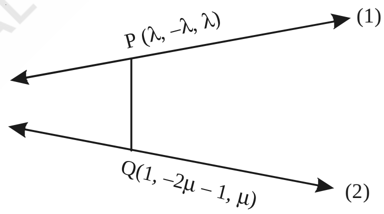
Ans. The equations of the given straight lines in Cartesian form are :

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{1} \quad ...(1)$$

and $$\frac{x-1}{0} = \frac{y+1}{-2} = \frac{z}{1} \quad ...(2)$$

Since, the direction-ratios are not proportional,

$\therefore$ the lines are not parallel.



Let P be a point on line (1) and Q be a point on line (2) such that PQ is parpendicular to both the straight lines.

Let P be $(\lambda, -\lambda, \lambda)$ and Q be $(1, -2\mu - 1, \mu)$, where 'λ' and 'λ' are scalars. Then, the direction-ratios of the line PQ are :

$$< \lambda - 1, -\lambda + 2\mu + 1, \lambda - \mu >$$

Since, PQ is perpendicular to lines (1) and (2),

$\therefore$ $(\lambda - 1)(1) + (-\lambda + 2\mu + 1)(-1) + (\lambda + \mu)(1) = 0$

and $(\lambda - 1)(0) + (-\lambda + 2\mu - 1)(-2) + (\lambda + \mu)(1) = 0$

$\Rightarrow \quad 3\lambda - 3\mu = 2 \quad ...(3)$

$\Rightarrow \quad 3\lambda - 5\mu = 2 \quad ...(4)$

Solving (3) and (4),

$$\lambda = \frac{2}{3},\ \mu = 0.$$

$\therefore$ Co ordinate of P are $\left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{2}\right)$ and those of Q are $(1, -1, 0)$.

Hence, required S.D.

$$= \sqrt{\left(1-\frac{2}{3}\right)^2 + \left(-1+\frac{2}{3}\right)^2 + \left(0-\frac{2}{3}\right)^2} = \sqrt{\frac{2}{3}} \text{ units.}$$

SECTION—D

(This section comprises of Long Answer Type questions (LAQ) of 5 marks each)

Q. 32. Find the area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Ans. The given ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$ ...(1)

Since (1) is symmetrical about both axes,

$\therefore$ area of the ellipse = 4 (Shaded area).

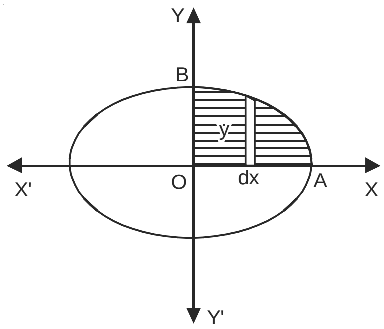


Fig.

= 4 (area OAB) ...(2)

But area OAB = $\int_0^4 y\,dx$ [*Taking vertical strips*]

$$= \int_0^4 \frac{3}{4}\sqrt{16-x^2}\,dx$$

[$\because$ $\frac{x^2}{16}+\frac{y^2}{9}=1 \Rightarrow \frac{y^2}{9}=1-\frac{x^2}{16}$

$\Rightarrow y = \frac{3}{4}\sqrt{16-x^2}$ ($\because y > 0$)]

$$= \frac{3}{4}\left[\frac{x\sqrt{16-x^2}}{2}+\frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= \frac{3}{4}\left[[2(0)+8\sin^{-1}(1)]-[0-0]\right] = \frac{3}{4}\left[8\frac{\pi}{2}\right] = 3\pi.$$

$\therefore$ From (2), area of the ellipse

= 4 (3π) = 12π sq. units.

Q. 33. Let f : X → Y be a function. Define a relation *R* in X given by :

R = {(*a*, *b*) : *f* (*a*) = *f* (*b*)}.

Examine, whether R is an equivalence relation or not.

Ans. For each $a \in X$, $(a, a) \in R$.

Thus R is *reflexive*. [$\because$ $f(a) = f(a)$]

Now $(a, b) \in R$ $\Rightarrow f(a) = f(b)$

$\Rightarrow f(b) = f(a)$

$\Rightarrow (b, a) \in R$.

Thus R is *symmetric.*

And $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow f(a) = f(b)$ and $f(b) = f(c)$

$\Rightarrow f(a) = f(c)$

$\Rightarrow (a, c) \in R$.

Thus R is *transitive.*

Hence, R is an equivalence relation.

Or

Show that the function $f : R_* \to R_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where R_* is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R_* ?

Ans. Do yourself.

Q. 34. Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

Ans. We have :

$$\vec{a}=3\hat{i}+2\hat{j}+2\hat{k} \text{ and } \vec{b}=\hat{i}+2\hat{j}-2\hat{k}.$$

$$\therefore \quad \vec{a}+\vec{b} = (3\hat{i}+2\hat{j}+2\hat{k})+(\hat{i}+2\hat{j}-2\hat{k}) = 4\hat{i}+4\hat{j}$$

$$\text{and} \quad \vec{a}-\vec{b} = (3\hat{i}+2\hat{j}+2\hat{k})-(\hat{i}+2\hat{j}-2\hat{k}) = 2\hat{i}+4\hat{k}.$$

$$\therefore (\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= (16-0)\hat{i}-(16-0)\hat{j}+(0-8)\hat{k}$$

$$= 16\hat{i}-16\hat{j}-8\hat{k}.$$

$\therefore$Unit vector perpendicular to both $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is given by :

$$\hat{n} = \pm\frac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}$$

$$= \pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{\sqrt{(16)^2+(-16)^2+(-8)^2}}$$

$$= \pm \frac{8(2\hat{i} - 2\hat{j} - \hat{k})}{8\sqrt{4+4+1}} = \pm \frac{2\hat{i} - 2\hat{j} - \hat{k}}{3}$$

$$= \pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k}.$$

Or

Find the angle between the following pairs of lines :

(*i*) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ **and**

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

(*ii*) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ **and**

$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$.

Ans. (*i*) The given pair of lines is :

$\vec{r} = (2\hat{i} - 5\hat{j} + \hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 2\hat{k})$ and

$\vec{r} = (7\hat{i} - 6\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$.

Here $\vec{b} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $b' = \hat{i} + 2\hat{j} + 2\hat{k}$,

where $|\vec{b}| = \sqrt{9+4+4} = \sqrt{17}$

and $|b'| = \sqrt{1+4+4} = \sqrt{9} = 3$

so that $\vec{b}.\vec{b'} = (3\hat{i} + 2\hat{j} + 2\hat{k}).(\hat{i} + 2\hat{j} + 2\hat{k})$

$= (3)(1) + (2)(2) + (2)(2)$

$= 3 + 4 + 4 = 11.$

If 'θ' be the required angle, then :

$$\cos\theta = \frac{\vec{b}.\vec{b'}}{|\vec{b}||\vec{b'}|} = \frac{11}{\sqrt{17}(3)} = \frac{11}{3\sqrt{17}}.$$

Hence, $\theta = \cos^{-1}\left(\frac{11}{3\sqrt{17}}\right)$.

(*ii*) Similar to part (*i*).

Q. 35. Two numbers are selected at random (without replacement) from positive numbers 2, 3, 4, 5, 6 and 7. Let X denote the larger of the two numbers obtained. Find the probability distribution of X.

Ans. Here, $n(S) = {}^6C_2 = \frac{6\times5}{2\times1} = 15$,

Let X be the larger of that two numbers obtained.

∴ X = 3, 4, 5, 6, 7.

∴ Probability distribution is :

X :	3	4	5	6	7
P(X) :	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

SECTION—E

(This section comprises of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36. Read the following passage and answer the question, which follows :

On her birthday, Seema decided to donate some money to children of an orophanage home. If there were 8 children less, everyone would have got ₹ 10 more. However, if there were 16 children more, everyone would have got v10 less. Let the number of children be *x* and the amount distributed by Seema for one child be *y* (in ₹).

Based on the above information, answer the following :

(I) The equation in terms of *x* and *y* are :

(*a*) $5x - 4y = 40$, $5x - 8y = -80$ (*b*) $5x - 4y = 40$, $5x - 8y = 80$

(*c*) $5x - 4y = 40$, $5x + 8y = -80$ (*d*) $5x + 4y = 40$, $5x - 8y = -8$.

(II) Which of the following matrix equations represent the information given above ?

(*a*) $\begin{bmatrix} 5 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ -80 \end{bmatrix}$ (*b*) $\begin{bmatrix} 5 & -4 \\ 5 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \end{bmatrix}$

(*c*) $\begin{bmatrix} 5 & -4 \\ 5 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ -80 \end{bmatrix}$ (*d*) $\begin{bmatrix} 5 & 4 \\ 5 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ -80 \end{bmatrix}$.

(III) the number of children who were given some money by Seema, is :

(*a*) 30 (*b*) 40

(*c*) 23 (*d*) 32.

(IV) How much amount is given to each child by Seema ?

(*a*) 32 (*b*) 30

(*c*) 62 (*d*) 26.

(V) How much amount Seema spends in distributing the money to all the students of the Orophanage ?

(*a*) 609 (*b*) 960

(*c*) 906 (*d*) 690.

Ans. (I) (*a*) **(II)** (*c*) **(III)** (*d*) **(IV)** (*b*) **(V)** (*b*).

Q. 37. In a residential society comprising of 100 houses, there were 60 children between the ages of 10–15 years. They were inspired by their teachers to start composting to ensure that biodegradable waste is recycled. For this purpose, instead of each child doing it for only his/her house, children convinced the Residents Welfare Association to do it as a society initiative. For this they identified a square area in the local park. Local authorities charged amount of ₹50 per square metre for space so that there is no misuse of the space and Residents welfare association takes it seriously. Association hired a labourer for digging out 250 m^3 and he charged ₹400 × $(depth)^2$. Association will like to have minimum cost.

Compost Pit

Based on the above information, answer the following:

(I) Let side of square plot be x m and its depth be h m, then find the cost c for the pit.

(II) Find the value of h (in m) for which $\frac{dc}{dh} = 0$.

(III) Find the value of x (in m) for minimum cost.

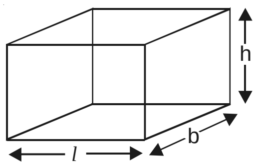
OR

Find the total minimum cost of digging the pit (in ₹)

Ans. (I) *Given*, volume of cuboid (pit) = 250 m^3

Digging cost = ₹ 400 × $(\text{depth})^2$

Let $l = x$ m, $b = x$ m and depth = h m.



$\therefore\ l \times b \times h = 250 \Rightarrow x \times x \times h = 250$

$\Rightarrow x^2 = \frac{250}{h}$ .....(1)

For 1 m^2, local authorities charged = ₹ 50
For x^2 m^2, local authorities charges = ₹ 50 x^2
For digging out, labourer charged = 400 h^2
$\therefore$ c, total cost of digging pit = 50 x^2 + 400 h^2

$= 50 \times \frac{250}{h} + 400h^2$ [*Using (1)*]

$= \frac{12500}{h} + 400\,h^2.$

(II) $\frac{dc}{dh} = 0$

$\therefore\ -\frac{12500}{h^2} + 800h = 0$

$\Rightarrow h^3 = \frac{125}{8} \Rightarrow h = \frac{5}{2} = 2.5$

(III) $\left.\frac{d^2c}{dh^2}\right]_{h=\frac{5}{2}} = \frac{25000}{\left(\frac{5}{2}\right)^3} + 800 > 0$.

Thus, $x = \frac{5}{2}$ is a point of minimum.

$\therefore\ x^2 = \frac{250}{5} \times 2 \Rightarrow x^2 = 100 \Rightarrow x = 10.$

Or

Minimum cost is at $h = \frac{5}{2}$.

$\therefore\ c\,(h) = \frac{12500}{h} + 400h^2$.

$\therefore\ c\left(\frac{5}{2}\right) = \frac{12500}{\frac{5}{2}} + 400 \times \frac{25}{4}$

$= \frac{12500}{5} \times 2 + 2500 = 5000 + 2500 =$ ₹ 7500.

Q. 38. In an office three employers Vinay, Sonia and Iqbal process incoming copies of a certain form. Vinay processes 50% of the forms. Sonia processes 20% and Iqbal the ramaining 30% of the forms. Vinay has an error rate of 0.06, Sonia error rate of 0.04 and Iqbal has an error rate of 0.03.

Based an the above information, answer the following :

(I) Find the conditional probability that an error is committed in process, given that Sonia processed the form.

(II) Find the probability that Sonia processed the form and committed an error.

(III) Find the total probability of committing an error in processing the form.

Or

Let A be the event of committing an error in processing the form and let E_1, E_2 and E_3 be the event that Vinay, Sonia and Iqbal processed the form. Find the value of $\sum_{i=1}^{3} P(E_1 / A)$.

Ans. (I) Reqd. probablity $= \frac{0.04}{1} = 0.04$.

(II) Reqd. probablity= $0.04 \times \frac{1}{5} = 0.008$.

(III) Reqd. probability

$= 0.06 \times \frac{5}{100} + 0.04 \times \frac{20}{100} + 0.13 \times \frac{30}{100} = 0.047.$

Or

P (E_1/A) + P (E_2/A) + P (E_3/A) = 1.

[*Total Probability*]

Holy Faith New Style Sample Paper–10

(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions : Same as Holy Faith New Style Sample Paper–1.

SECTION—A

(Multiple choice questions, each question carries 1 mark)

1. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 3x$. Then :

(*a*) f is one–one onto

(*b*) f is many–one onto

(*c*) f is one–one but not onto

(*d*) f is neither one–one nor onto.

Ans. (*a*) f is one–one onto.

2. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ is equal to :

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{3}$

(*c*) $\frac{1}{4}$ (*d*) 1.

Ans. (*d*) 1.

3. For the matrix $X = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $(X^2 - X)$ is :

(*a*) 2I (*b*) 3I

(*c*) I (*d*) 5I.

Ans. (*d*) 5I.

4. If $\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$, then value of : $a + b - c + 2d$ is :

(*a*) 8 (*b*) 10

(*c*) 4 (*d*) – 8.

Ans. (*d*) – 8.

5. If $\begin{vmatrix} \alpha & 3 & 4 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 0$, then the value of α is :

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4.

Ans. (*d*) 4.

6. The function $f(x) = \begin{cases} \dfrac{e^{3x} - e^{-5x}}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$

is continuous at $x = 0$ for the value of k, as :

(*a*) 3 (*b*) 5

(*c*) 2 (*d*) 8.

Ans. (*d*) 8.

7. If $y = 5 \cos x - 3 \sin x$, then $\frac{d^2y}{dx^2}$ is equal to :

(*a*) $-y$ (*b*) y

(*c*) $25y$ (*d*) $9y$.

Ans. (*a*) $-y$.

8. Function $f(x) = 2x + 3$ is :

(*a*) Strictly increasing on **R**

(*b*) Strictly decreasing on **R**

(*c*) Neither increasing nor decreasing on **R**

(*d*) A discontinuus function.

Ans. (*a*) Strictly increasing on **R.**

9. Which of these is equal to $\int e^{(x \log 5)} e^x \, dx$, c is the constant of integration ?

(*a*) $\frac{(5e)^x}{\log 5e} + c$ (*b*) $\log 5^x + x + c$

(*c*) $5^x e^x + c$ (*d*) $(5e)^x \log x + c$.

Ans. (*a*) $\frac{(5e)^x}{\log 5e} + c$.

10. $\int_{-1}^{e} \frac{\log x}{x} dx$ is equal to :

(*a*) $\frac{e^2}{2}$ (*b*) 1

(*c*) $\frac{1}{2}$ (*d*) $-\infty$.

Ans. (*c*) $\frac{1}{2}$.

11. The area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$ and $x = 4$ is (in sq. units):

(*a*) $\frac{15}{2}$ (*b*) $\frac{14}{3}$

(*c*) 7 (*d*) None of these.

Ans. (*b*) $\frac{14}{3}$.

12. Which of the following is a homogeneous differential equation ?

(*a*) $(4x + 6y + 5)\, dy - (3y + 2x + 4)\, dx = 0$

(*b*) $xy\, dx - (x^3 + y^3)\, dy = 0$

(*c*) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

(*d*) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$.

Ans. (*d*) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$.

13. The solution of the differential equation : $x\frac{dy}{dx} + 2y = x^2$ is :

(*a*) $y = \frac{x^2 + c}{4x^2}$ (*b*) $y = \frac{x^2}{4} + c$

(*c*) $y = \frac{x^4 + c}{x^2}$ (*d*) $y = \frac{x^4 + c}{4x^2}$.

Ans. (*d*) $y = \frac{x^4 + c}{4x^2}$.

14. If the vector $\hat{i} + b\hat{j} + \hat{k}$ is equally inclined to the coordinate axes, then the value of '*b*' is:

(*a*) –1 (*b*) 1

(*c*) $-\sqrt{3}$ (*d*) $-\frac{1}{\sqrt{3}}$.

Ans. (*b*) 1.

15. If $|\vec{a}| = 4$ and $-3 \le \lambda \le 2$, then $|\lambda \vec{a}|$ lies in :

(*a*) [0, 12] (*b*) [2, 3]

(*c*) [8,12] (*d*) [–12, 8].

Ans. (*d*) [–12, 8].

16. The direction-ratios of a line parallel to *z*-axis are :

(*a*) $< 1, 1, 0 >$ (*b*) $< 1, 1, 1 >$

(*c*) $< 0, 0, 0 >$ (*d*) $< 0, 0, 1 >$.

Ans. (*d*) $< 0, 0, 1 >$.

17. The corner points of the feasible region determined by a set of constraints (linear inequalities) are P (0, 5), Q (3, 5), R (5, 0) and S (4, 1) and the objective function is $Z = ax + 2by$, where a, $b > 0$. The condition on a and b such that the maximum Z occurs at Q and S is :

(*a*) $a - 5b = 0$ (*b*) $a - 3b = 0$

(*c*) $a - 2b = 0$ (*d*) $a - 8b = 0$.

Ans. (*d*) $a - 8b = 0$.

18. A coin is tossed and then a die is thrown. What is the probability of obtaining a 6, given a head came up ?

(*a*) $\frac{1}{3}$ (*b*) $\frac{1}{6}$

(*c*) $\frac{4}{6}$ (*d*) $\frac{1}{12}$.

Ans. (*d*) $\frac{1}{12}$.

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.

(*b*) Both 'A' and 'R' are true but 'R' is not the correct explanation of 'A'.

(*c*) 'A' is true but 'R' is false.

(*d*) 'A' is false but 'R' is true.

19. **Assertion (A) :** If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$.

Reason (R) : The inverse of a diagonal metrix, is a diagonal matrix.

Ans. (*b*) Both 'A' and 'R' are true but 'R' is not the correct explanation of 'A'.

20. Let a solution $y = y(x)$ of the differential equation : $x\sqrt{x^2 - 1}\, dy - y\sqrt{y^2 - 1}\, dx = 0$ satisfy $f(2) = \frac{2}{\sqrt{3}}$.

Assertion–(A) : $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$ and

Reason-(R) : $y(x)$ is given by :

$$\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}.$$

Ans. (*c*) 'A' is true but 'R' is false.

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Prove that :

$$\tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}.$$

Ans. Put $\cos^{-1}\frac{a}{b} = \theta$ so that $\cos\theta = \frac{a}{b}$.

$$\text{LHS} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$= \frac{\left(1+\tan\frac{\theta}{2}\right)^2 + \left(1-\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{2(1+\tan^2\theta)}{1-\tan^2\frac{\theta}{2}} = \frac{2}{\cos 2.\frac{\theta}{2}} = \frac{2}{\cos\theta}$$

$$= \frac{2}{a/b} = \frac{2b}{a} = \textbf{RHS.}$$

Or

If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$, then find '$x$'.

Ans. We have : $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$ ...(1)

Put $\tan^{-1} x = t$ so that $\cot^{-1} x = \frac{\pi}{2} - t$.

$\therefore$ (1) becomes : $t^2 + \left(\frac{\pi}{2} - t\right)^2 = \frac{5\pi^2}{8}$

$\Rightarrow t^2 + \frac{\pi^2}{4} - \pi t + t^2 = \frac{5\pi^2}{8}$

$\Rightarrow 2t^2 - \pi t - \frac{3\pi^2}{8} = 0$

$\Rightarrow 16t^2 - 8\pi t - 3\pi^2 = 0$.

Solving, $t = \frac{8\pi \pm \sqrt{64\pi^2 + 192\pi^2}}{32}$

$= \frac{8\pi \pm 16\pi}{32} = \frac{3\pi}{4}, -\frac{\pi}{4}$.

When $\tan^{-1} x = \frac{3\pi}{4}$, then $x = \tan\frac{3\pi}{4} = -1$.

When $\tan^{-1} x = -\frac{\pi}{4}$,

then $x = \tan\left(-\frac{\pi}{4}\right) = -1$.

Hence, $x = -1$.

Q. 22. Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function of 'x' throughout its domain.

Ans. Let $f(x) = \log(1+x) - \frac{2x}{2+x}$.

$\therefore f'(x) = \frac{1}{1+x} - 2\left[\frac{(2+x)(1) - x(0+1)}{(2+x)^2}\right]$

$= \frac{1}{1+x} - 2\left[\frac{2}{(2+x)^2}\right]$

$= \frac{(x+2)^2 - 4(x+1)}{(x+1)(x+2)^2} = \frac{x^2}{(x+1)(x+2)^2}$

> 0

$\left[\because \frac{x^2}{(x+2)^2}, \textit{being a perfect sq., is } +ve \textit{ and } x > -1 \Rightarrow x+1 > 0\right]$

Hence, 'f' is an increasing function of x throughout its domain.

Q. 23. Determine the absolute maximum and absolute minimum values of each of the following in the stated domains :

Determine the absolute maximum and absolute minimum value of

$$f(x) = (x+1)^{2/3}\ ;\ 0 \le x \le 8.$$

Ans. We have : $f(x) = (x+1)^{2/3}$ ...(1)

$\therefore f'(x) = \frac{2}{3}(x+1)^{-1/3}$.

But $f'(x) = 0$ gives no value in $0 \le x \le 8$.

So we are to examine the max./min. at the end points only.

Now $f(0) = (0+1)^{2/3} = 1$

and $f(8) = (8+1)^{2/3} = 9^{2/3} = \sqrt[3]{81} = 3\sqrt[3]{3}$.

Hence, the absolute maximum value of $f(x) = 3\sqrt[3]{3}$ and absolute minimum value of $f(x) = 1$.

Or

Find the points of local maxima and local minima, if any, of the function :

$$f(x) = (x-1)(x+2)^2.$$

Find also the local maximum and local minimum values.

Ans. We have : $f(x) = (x-1)(x+2)^2$.

$\therefore f'(x) = (x-1)\,.\,\frac{d}{dx}(x+2)^2 + (x+2)^2\,.\,\frac{d}{dx}(x-1)$

$= (x-1) \times 2(x+2) + (x^2 + 4x + 4) \times 1$

$= 2(x^2 + x - 2) + x^2 + 4x + 4$

$= 3x^2 + 6x = 3x(x+2)$.

Now $f'(x) = 0$

$\Rightarrow 3x(x+2) = 0$

$\Rightarrow x = 0, -2$

and $f''(x) = 6x + 6 = 6(x+1)$ ...(1)

Now $f''(x)\big]_{x=0} = 6(0+1) = 6$ (+ ve)

and $f''(x)\big]_{x=-2} = 6(-2+1) = -6$ (– ve)

Hence, $f(x)$ has local max. at $x = -2$ and local min. at $x = 0$.

Also, $f(-2) = (-2-1)(-2+2)^2 = (-3)(0) = 0$, local max. value

and $f(0) = (0-1)(0+2)^2 = (-1)(4) = -4$, local min. value.

Aliter. Upto (1), repeat as above.

At $x = 0$. When x is slightly less than 0, then $f'(x)$ is – ve.

When x is slightly greater than 0, then $f'(x)$ is + ve.

Thus $f'(x)$ changes sign from –ve to +ve.

So $x = 0$ is a point of local minima and local minimum value $= f(0) = (0-1)(0+2)^2 = -4$.

At $x = -2$. When x is slightly less than –2, then $f'(x)$ is +ve.

When x is slightly greater than –2, then $f'(x)$ is –ve.

Thus $f'(x)$ changes sign from +ve to –ve.

So $x = -2$ is a point of local maxima and local maximum value

$$= f(-2) = (-2-1)(-2+2)^2$$
$$= (-3)(0)^2 = (-3)(0) = 0.$$

Q. 24. Evaluate : $\displaystyle\int_0^{\pi/4} \frac{dx}{1+\tan x}$.

Ans. $$\frac{1}{1+\tan x} = \frac{1}{1+\frac{\sin x}{\cos x}} = \frac{\cos x}{\cos x + \sin x}$$

$$= \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2(\cos x + \sin x)}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{\cos x - \sin x}{\cos x + \sin x}.$$

$$\therefore \int_0^{\pi/4} \frac{1}{1+\tan x} dx$$

$$= \left[\frac{1}{2}x\right]_0^{\pi/4} + \frac{1}{2}\left[\log|\cos x + \sin x|\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] + \frac{1}{2}\left[\log\left|\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right| - \log|1+0|\right]$$

$$= \frac{\pi}{8} + \frac{1}{2}[\log\sqrt{2} - \log 1]$$

$$= \frac{\pi}{8} + \frac{1}{4}\log 2 - 0 = \frac{\pi}{8} + \frac{1}{4}\log 2.$$

Q. 25. Find $\frac{dy}{dx}$ if $x^{2/3} + y^{2/3} = a^{2/3}$

Ans. The given equation is $x^{2/3} + y^{2/3} = a^{2/3}$ ... (1)

Its parametric equations are :

$x = a\cos^3\theta$, $y = a\sin^3\theta$ ...(2)

$$\left[\because x^{2/3} + y^{2/3} = a^{2/3}(\cos^2\theta + \sin^2\theta) = a^{2/3}(1) = a^{2/3}\right]$$

$$\therefore \frac{dx}{d\theta} = -3a\cos^2\theta \sin\theta \text{ and } \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta.$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$= -\frac{\sin\theta}{\cos\theta} = -\frac{\sqrt[3]{\frac{y}{a}}}{\sqrt[3]{\frac{x}{a}}} \quad \text{[Using (2)]}$$

$$= -\sqrt[3]{\frac{y}{x}}.$$

SECTION—C

(This section comprises of short answer type questions (SAQ) of 3 marks each)

Q. 26. Integrate the rational function $\frac{3x-1}{(x-1)(x-2)(x-3)}$.

Ans. Let $$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

$$\equiv \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \quad ...(1)$$

Multiplying by $(x-1)(x-2)(x-3)$, we get :

$3x - 1 \equiv A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$

Putting $x = 1$, $2 = A(-1)(-2)$ ⇒ A = 1.

Putting $x = 2$, $5 = B(1)(-1)$ ⇒ B = – 5.

Putting $x = 3$, $8 = C(2)(1)$ ⇒ C = 4.

∴ From (1),

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} + \frac{-5}{x-2} + \frac{4}{x-3}.$$

$$\therefore \int \frac{3x-1}{(x-1)(x-2)(x-3)} dx$$

$$= \int \frac{1}{x-1} dx - 5\int \frac{1}{x-2} dx + 4\int \frac{1}{x-3} dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + c.$$

Q. 27. In a hostel, 60% of the students read Hindi news paper, 40% read English news paper and 20% read both Hindi and English news papers. A student is selected at random.

(*a*) Find the probability that she reads neither Hindi nor English news papers.

(*b*) If she reads Hindi news paper, find the probability that she reads English news paper.

(*c*) If she reads English news paper, find the probability that she reads Hindi news paper.

Ans. We have : $P(H) = \frac{60}{100} = \frac{3}{5}$,

$$P(E) = \frac{40}{100} = \frac{2}{5}$$

and $P(H \cap E) = \frac{20}{100} = \frac{1}{5}$.

(*a*) $P(H \cup E) = P(H) + P(E) - P(H \cap E)$

$$= \frac{3}{5} + \frac{2}{5} - \frac{1}{5} = \frac{4}{5}.$$

$\therefore$ P (neither Hindi nor English) $= 1 - P(H \cup E)$

$$= 1 - \frac{4}{5} = \frac{1}{5}.$$

(*b*) $P(E/H) = \frac{P(E \cap H)}{P(H)} = \frac{1/5}{3/5} = \frac{1}{3}.$

(*c*) $P(H/E) = \frac{P(E \cap H)}{P(E)} = \frac{1/5}{2/5} = \frac{1}{2}.$

Q. 28. Find : $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}\, d\theta.$

Ans. $I = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}\, d\theta$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}\, d\theta.$$

Put $\sin\theta = t$ so that $\cos\theta\, d\theta = dt$.

$$\therefore \quad I = \int \frac{dt}{(4+t^2)(1+4t^2)} \quad ...(1)$$

Now $\frac{1}{(4+t^2)(1+4t^2)} = \frac{1}{(4+x)(1+4x)}$, where $t^2 = x$

$$= \frac{1}{(4+x)(1+4(-4))} + \frac{1}{\left(4-\frac{1}{4}\right)(1+4x)}$$

$$= -\frac{1}{15(4+x)} + \frac{1}{\frac{15}{4}(1+4x)} \quad [\textit{Partial Fractions}]$$

$$= -\frac{1}{15(4+t^2)} + \frac{4}{15(1+4t^2)}.$$

From (1), $I = -\frac{1}{15}\int \frac{1}{2^2+t^2}\, dt + \frac{4}{15} \times \frac{1}{4}\int \frac{1}{\frac{1}{4}+t^2}\, dt$

$$= -\frac{1}{15} \cdot \frac{1}{2}\tan^{-1}\frac{t}{2} + \frac{1}{15} \times \frac{1}{1/2}\tan^{-1}\frac{t}{1/2} + c$$

$$= -\frac{1}{30}\tan^{-1}\frac{\sin\theta}{2} + \frac{2}{15}\tan^{-1}(2\sin\theta) + c.$$

Or

Evaluate the integral :

$$\int_0^{\pi/2} \frac{\sin\theta}{(25+\cos\theta)(26+\cos\theta)}\, d\theta.$$

Show your steps.

Ans. Let $I = \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{(25+\cos\theta)(26+\cos\theta)}$.

Put $\cos\theta = t$ so that $-\sin\theta\, d\theta = dt$

i.e. $\sin\theta\, d\theta = -dt$.

When $\theta = 0$, $t = \cos 0 = 1$; when $\theta = \frac{\pi}{2}$, $t = \cos\frac{\pi}{2} = 0$.

$$\therefore \quad I = \int_1^0 \frac{-dt}{(25+t)(26+t)}$$

$$= \int_0^1 \frac{dt}{(t+25)(t+26)} \quad [\textit{Property II}]$$

$$= \int_0^1 \left(\frac{1}{t+25} - \frac{1}{t+26}\right) dx$$

[*Partial Fractions*]

$$= \int_0^1 \frac{1}{t+25}\, dt - \int_0^1 \frac{1}{t+26}\, dt$$

$$= [\log|t+25|]_0^1 - [\log|t+26|]_0^1$$

$$= (\log 26 - \log 25) - (\log 27 - \log 26)$$

$$= 2\log 26 - (\log 25 + \log 27)$$

$$= \log 26^2 - \log 25 \times 27 = \log\frac{26^2}{25 \times 27}$$

$$= \log\left(\frac{676}{675}\right).$$

Q. 29. In a bank, principal increases continuously at the rate of r% per year. Find the value of r if ₹ 100 double itself in 10 years ($\log_e 2 = 0.6931$).

Ans. Let 'P' be the principal at any time t.

By the question, $\frac{dP}{dt} = \frac{r}{100} \times P$

$$\Rightarrow \quad \frac{dP}{P} = \frac{r}{100} \times dt. \quad |\textit{Variables Separable}$$

Integrating, $\int \frac{dP}{P} = \frac{r}{100}\int 1.dt + c$

$\Rightarrow$ $\log_e P = \frac{rt}{100} + c$ ...(1) [$\because P > 0$]

When $t = 0$, P = 100, $\therefore \log_e 100 = c$

Putting in (1), $\log_e P = \frac{rt}{100} + \log_e 100$ ...(2)

When $t = 10$, P = 200, $\log_e 200 = \frac{10\,r}{100} + \log_e 100$

$\Rightarrow$ $\log_e \frac{200}{100} = \frac{r}{10}$ $\Rightarrow r = 10 \log_e 2 = 10\,(0.6931) = 6.931.$

Hence, $r = 6.931\%$.

Or

Solve $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0;\ y(0) = 0$

Ans. The given equation can be written as:

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2} \quad ...(1)\ |\ \textit{Linear Equation}$$

Comparing with $\frac{dy}{dx} + Py = Q$, we have:

'P' $= \frac{2x}{1+x^2}$ and 'Q' $= \frac{4x^2}{1+x^2}$.

$\therefore$ I.F. $= e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log|1+x^2|}$

$= e^{\log(1+x^2)} = (1 + x^2)$.

Multiplying (1) by $(1 + x^2)$, we get:

$$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$$

Intergrating, $y \cdot (1+ x^2) = \int 4x^2 dx = \frac{4x^3}{3} + c$

$\Rightarrow$ $y(1 + x^2) = \frac{4}{3}x^3 + c$ ...(2)

When $x = 0$, $y = 0$;

$\therefore$ $c = 0 + c$

$\Rightarrow$ $c = 0$.

Putting in (2), $y\,(1 + x^2) = \frac{4}{3}x^3$,

which is the required solution.

Q. 30. Verify that the function :

$y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx$,

where C_1, C_2 are arbitrary constants, is a solution of the differential equation :

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)\,y = 0.$$

Ans. The given equation is :

$y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx$ ...(1)

Diff. w.r.t. x, $\frac{dy}{dx} = e^{ax}\,[-b\,C_1 \sin bx + b\,C_2 \cos bx]$

$+ [C_1 \cos bx + C_2 \sin bx]\,e^{ax}.\,a$

$\Rightarrow \frac{dy}{dx} = e^{ax}\,[(b\,C_2 + a\,C_1) \cos bx + (a\,C_2 - b\,C_1) \sin bx]$

...(2)

Again diff. w.r.t. x,

$\frac{d^2y}{dx^2} = e^{ax}\,[(b\,C_2 + a\,C_1)(-b \sin bx) + (a\,C_2 - b\,C_1)$

$(b \cos bx)] + [(bC_2 + aC_1) \cos bx + (aC_2 - bC_1) \sin bx)].\,ae^{ax}$

$= e^{ax}\,[(a^2 C_2 - 2ab\,C_1 - b^2 C_2) \sin bx$

$+ (a^2C_1 + 2ab\,C_2 - b^2 C_1) \cos bx]$ ...(3)

LHS $= \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)\,y$

$= e^{ax}\,[(a^2 C_2 - 2ab\,C_1 - b^2 C_2) \sin bx$

$+ (a^2 C_1 + 2ab\,C_2 - b^2 C_1) \cos bx]$

$- 2a\,e^{ax}[(bC_2 + aC_1) \cos bx + (a\,C_2 - b\,C_1) \sin bx]$

$+ (a^2 + b^2)\,[C_1 \cos bx + C_2 \sin bx]\,e^{ax}$

$= e^{ax}\,[(a^2 C_2 - 2ab\,C_1 - b^2 C_2 - 2a^2 C_2$

$+ 2ab\,C_1 + a^2 C_2 + b^2 C_2) \sin bx$

$+ (a^2 C_1 + 2ab\,C_2 - b^2C_1 - 2ab\,C_2 - 2a^2 C_1 + a^2 C_1$

$+ b^2 C_1) \cos bx]$

$= e^{ax}\,[(0) \sin bx + (0) \cos bx]$

$= 0 =$ **RHS.**

Hence, the verification.

Or

Using the matrix method, solve the following system of linear equations:

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4,\quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1,\quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Ans. The given system of equations can be written as :

$$AX = B \quad ...(1),$$

where $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, $X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$

$= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$

$= 2(75) - 3(-110) + 10(72)$

$= 150 + 330 + 720$

$= 1200 \neq 0$

$\Rightarrow$ A is non-singular $\Rightarrow A^{-1}$ exists.

Now adj $A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}'$ **[Do it]**

$= \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$.

Hence, $A^{-1} = \dfrac{\text{adj } A}{|A|}$

$= \dfrac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

From (1), $X = A^{-1}B$

$\Rightarrow \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} = \dfrac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$= \dfrac{1}{1200}\begin{bmatrix} 300 & +150 & +150 \\ 440 & -100 & +60 \\ 288 & 0 & -48 \end{bmatrix}$

$= \dfrac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$.

Comparing, $\dfrac{1}{x} = \dfrac{1}{2}, \dfrac{1}{y} = \dfrac{1}{3}$ and $\dfrac{1}{z} = \dfrac{1}{5}$.

Hence, $x = 2$, $y = 3$ and $z = 5$.

Q. 31. Maximise $Z = -x + 2y$, subject to the constraints :

$x \geq 3, x + y \geq 5, x + 2y \geq 6, y \geq 0$.

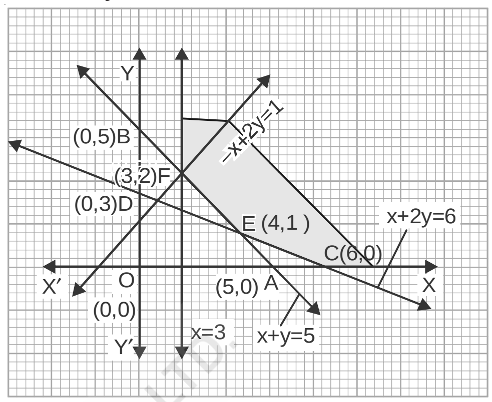
Ans. The system of constraints is :

$x \geq 3$ ...(1)

$x + y \geq 5$ ...(2)

$x + 2y \geq 6$ ...(3)

and $y \geq 0$ ...(4)



The shaded region in the above figure is the feasible region determined by (1)–(4).

The corner points are C (6, 0), E(4, 1) and F (3, 2).

Applying ***Corner Point Method,*** we have :

Corner Point	Corresponding Value of Z
C : (6, 0)	– 6
E : (4, 1)	– 2
F : (3, 2)	1

It appears that $Z_{max} = 1$ at F (3, 2).

But the feasible region is unbounded, therefore, we draw the graph of the inequation $-x + 2y > 1$.

Since the half-plane represented by $-x + 2y > 1$ has points common with the feasible region,

$\therefore \quad Z_{max} \neq 1$.

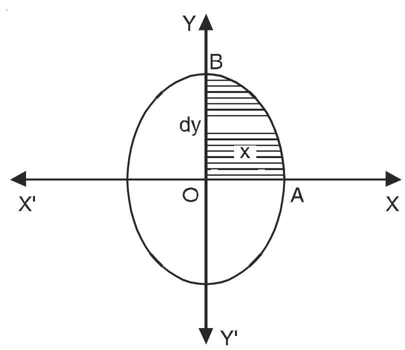
Hence, Z has no maximum value.

SECTION—D

(This section comprises of Long Answer Type questions (LAQ) of 5 marks each)

Q. 32. Find the area of the region bounded by the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

Ans.



The given ellipse is $\frac{x^2}{4}+\frac{y^2}{9} = 1$ ...(1)

Since (1) is symmetrical about both the axes,

$\therefore$ area of the ellipse = 4 (Shaded area)

= 4 (area OAB) ...(2)

But area OAB = $\int_0^3 x\, dy$

[*Taking horizontal strips*]

$$= \int_0^3 \frac{2}{3}\sqrt{9-y^2}\, dy$$

$$\left[\because \frac{x^2}{4}+\frac{y^2}{9} = 1 \Rightarrow \frac{x^2}{4} = 1-\frac{y^2}{9}\right.$$

$$\left.\Rightarrow x = \frac{2}{3}\sqrt{9-y^2} \quad (\because x > 0)\right]$$

$$= \frac{2}{3}\left[\frac{y\sqrt{9-y^2}}{2}+\frac{9}{2}\sin^{-1}\frac{y}{3}\right]_0^3$$

$$= \frac{2}{3}\left[\left[\frac{3}{2}(0)+\frac{9}{2}\sin^{-1}(1)\right]-[0-0]\right]$$

$$= \frac{2}{3}\left[\frac{9}{2}\left(\frac{\pi}{2}\right)\right] = \frac{3\pi}{2}.$$

$\therefore$ From (2), area of the ellipse = $4\left(\frac{3\pi}{2}\right) = 6\pi$ sq. units.

Q. 33. If R_1 and R_2 are equivalence relations in a set A ($\neq \phi$), show that $R_1 \cap R_2$ is also an equivalence relation.

Ans. Since R_1 and R_2 are equivalence relations, [*Given*]

$\therefore (a, a) \in R_1$ and $(a, a) \in R_2 \ \forall \ a \in A$

$\Rightarrow (a, a) \in R_1 \cap R_2 \ \forall \ a \in A$.

Thus $R_1 \cap R_2$ is *reflexive.*

Now $(a, b) \in R_1 \cap R_2$

$\Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2$

$\Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2$

$\Rightarrow (b, a) \in R_1 \cap R_2$.

Thus $R_1 \cap R_2$ is *symmetric.*

And $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$

$\Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2$

$\Rightarrow (a, c) \in R_1 \cap R_2$.

Thus $R_1 \cap R_2$ is *transitive.*

Hence, $R_1 \cap R_2$ is an equivalence relation.

Or

Show that the function : $f : \mathbf{N} \to \mathbf{N}$ given by f(1) = f(2) = 1 and f (x) = x – 1, for every x > 2 is onto but not one-one.

Ans. Since $f(1) = f(2) = 1$,

$\therefore \quad f(1) = f(2)$,

where $1 \neq 2$.

$\therefore$ 'f' is not one–one.

Suppose there exists $x_0 \in \mathbf{N}$

such that $f(x_0) = y$

i.e., $x_0 - 1 = y \Rightarrow x_0 = y + 1$.

Also $1 \in \mathbf{N}$, $f(1) = 1$.

Thus 'f' is onto.

Hence, 'f' is onto but not one–one.

Q. 34. Find 'λ' and 'μ' if

$$\mathbf{(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k}) = \vec{0}.}$$

Ans. Let $\vec{a} = 2\hat{i}+6\hat{j}+27\hat{k}$ and $\vec{b} = \hat{i}+\lambda\hat{j}+\mu\hat{k}$.

$$\therefore \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix}$$

$= \hat{i}(6\mu - 27\lambda) - \hat{j}(2\mu - 27) + \hat{k}(2\lambda - 6)$

$= (6\mu - 27\lambda)\hat{i} + (27 - 2\mu)\hat{j} + (2\lambda - 6)\hat{k}$.

By the question, $\vec{a}\times\vec{b} = \vec{0}$

$\Rightarrow (6\mu - 27\lambda)\hat{i} + (27 - 2\mu)\hat{j} + (2\lambda - 6)\hat{k} = 0$

$\Rightarrow 6\mu - 27\lambda = 0$ ...(1) $27 - 2\mu = 0$ ...(2)

and $(2\lambda - 6) = 0$ ...(3)

From (3), $\lambda = 3$. From (2), $\mu = \frac{27}{2}$.

These satisfy (1). $\left[\because 6\left(\frac{27}{2}\right) - 27(3) = 0\right]$

Hence, $\lambda = 3$ and $\mu = \frac{27}{2}$.

Or

Find the angle between the following pair of lines :

(*i*) $\mathbf{\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}}$ **and** $\mathbf{\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}}$.

(*ii*) $\mathbf{\frac{x}{2} = \frac{y}{2} = \frac{z}{1}}$ **and** $\mathbf{\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}}$.

Ans. (*i*) The given lines are parallel to the vectors :

$$\vec{b_1} = 2\hat{i}+5\hat{j}-3\hat{k}$$

and $\vec{b_2} = -\hat{i}+8\hat{j}+4\hat{k}$.

If 'θ' be the angle between the given lines,

then $\cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$

$$= \frac{\left(2\hat{i}+5\hat{j}-3\hat{k}\right).\left(-\hat{i}+8\hat{j}+4\hat{k}\right)}{\left|2\hat{i}+5\hat{j}-3\hat{k}\right|\left|-\hat{i}+8\hat{j}+4\hat{k}\right|}$$

$$= \frac{(2)(-1)+(5)(8)+(-3)(4)}{\sqrt{4+25+9}\sqrt{1+64+16}}$$

$$= \frac{-2+40-12}{\sqrt{38}\sqrt{81}} = \frac{26}{\sqrt{38}(9)} = \frac{26}{9\sqrt{38}}.$$

Hence, $\theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$.

(*ii*) The given lines are parrallel to the vectors :

$$\vec{b_1} = 2\hat{i}+2\hat{j}+\hat{k}$$

and $\vec{b_2} = 4\hat{i}+\hat{j}+8\hat{k}$.

If 'θ' be the angle between the given lines,

then $\cos\theta = \frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|}$

$$= \frac{\left(2\hat{i}+2\hat{j}+\hat{k}\right).\left(4\hat{i}+\hat{j}+8\hat{k}\right)}{\sqrt{4+4+1}\sqrt{16+1+64}}$$

$$= \frac{(2)(4)+(2)(1)+(1)(8)}{\sqrt{9}\sqrt{81}}$$

$$= \frac{8+2+8}{(3)(9)} = \frac{18}{(3)(9)} = \frac{18}{27} = \frac{2}{3}.$$

Hence, $\theta = \cos^{-1}\left(\frac{2}{3}\right)$.

Q. 35. A girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin two times and notes the number of heads obtained. If she obtained exactly two heads, what is the probability that she throws 1, 2, 3 or 4 with the die.

Ans. Let the events be :

E_1 : The girl gets 1, 2, 3 and 4 on the die

E_2 : The girl gets 5 or 6 on the die and

A : Exactly two heads show up.

$\therefore \quad P(E_1) = \frac{4}{6} = \frac{2}{3}, P(E_2) = \frac{2}{6} = \frac{1}{3}.$

Also $P(A/E_1)$ = P (Exactly two heads show up when a coin is tossed two times)

$= P(\{HH\}) = \frac{1}{4}$

$P(E_2)$ = P (exactly two heads show up when a coin is tossed 3 times)

$= P(\{HHT, HTH, THH\}) = \frac{3}{8}.$

By *Bayes' Theorem,*

$$P(A/E_1) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$= \frac{\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)}{\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)+\left(\frac{1}{3}\right)\left(\frac{3}{8}\right)} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{8}}$$

$$= \frac{1}{6}\times\frac{24}{4+3} = \frac{4}{7}.$$

SECTION—E

(This section comprises of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36. Read the following passage and answer the question, which follows :

On her birthday, Seema decided to donate some money to children of an orophanage home. If there were 8 children less, everyone would have got ₹ 10 more. However, if there were 16 children more, everyone would have got v10 less. Let the number of children be *x* and the amount distributed by Seema for one child be *y* (in ₹).

Based on the above information, answer the following :

(I) The equation in terms of *x* and *y* are :

(*a*) $5x - 4y = 40$, $5x - 8y = -80$

(*b*) $5x - 4y = 40$, $5x - 8y = 80$

(*c*) $5x - 4y = 40$, $5x + 8y = -80$

(*d*) $5x + 4y = 40$, $5x - 8y = -8$.

(II) Which of the following matrix equations represent the information given above ?

(*a*) $\begin{bmatrix} 5 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ -80 \end{bmatrix}$

(*b*) $\begin{bmatrix} 5 & -4 \\ 5 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \end{bmatrix}$

(*c*) $\begin{bmatrix} 5 & -4 \\ 5 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ -80 \end{bmatrix}$

(*d*) $\begin{bmatrix} 5 & 4 \\ 5 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ -80 \end{bmatrix}$.

(III) the number of children who were given some money by Seema, is :

(*a*) 30 (*b*) 40

(*c*) 23 (*d*) 32.

(IV) How much amount is given to each child by Seema ?

(*a*) 32 (*b*) 30
(*c*) 62 (*d*) 26.

(V) How much amount Seema spends in distributing the money to all the students of the Orophanage ?

(*a*) 609 (*b*) 960
(*c*) 906 (*d*) 690.

Ans. (I) (*a*) **(II)** (*c*) **(III)** (*d*) **(IV)** (*b*) **(V)** (*b*).

Q. 37. In a residential society comprising of 100 houses, there were 60 children between the ages of 10–15 years. They were inspired by their teachers to start composting to ensure that biodegradable waste is recycled. For this purpose, instead of each child doing it for only his/her house, children convinced the Residents Welfare Association to do it as a society initiative. For this they identified a square area in the local park. Local authorities charged amount of ₹50 per square metre for space so that there is no misuse of the space and Residents welfare association takes it seriously. Association hired a labourer for digging out 250 m^3 and he charged ₹400 × (depth)^2. Association will like to have minimum cost.

Compost Pit

Based on the above information, answer the following:

(I) Let side of square plot be *x* m and its depth be *h* m, then find the cost *c* for the pit.

(II) Find the value of *h* (in m) for which $\frac{dc}{dh} = 0$.

(III) Find the value of *x* (in m) for minimum cost.

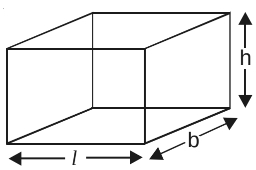
Or

Find the total minimum cost of digging the pit (in ₹)

Ans. (I) *Given*, volume of cuboid (pit) = 250 m^3

Digging cost = ₹ 400 × (depth)^2

Let $l = x$ m, $b = x$ m and depth = h m.



$\therefore\ l \times b \times h = 250 \Rightarrow x \times x \times h = 250$

$$\Rightarrow x^2 = \frac{250}{h} \qquad .....(1)$$

For 1 m^2, local authorities charged = ₹ 50

For x^2 m^2, local authorities charges = ₹ 50 x^2

For digging out, labourer charged = 400 h^2

$\therefore$ c, total cost of digging pit = $50x^2 + 400h^2$

$$= 50 \times \frac{250}{h} + 400h^2 \qquad [Using\ (1)]$$

$$= \frac{12500}{h} + 400h^2.$$

(II) $\frac{dc}{dh} = 0$

$$\therefore -\frac{12500}{h^2} + 800h = 0$$

$$\Rightarrow h^3 = \frac{125}{8} \Rightarrow h = \frac{5}{2} = 2.5$$

(III) $$\left.\frac{d^2c}{dh^2}\right]_{h=\frac{5}{2}} = \frac{25000}{\left(\frac{5}{2}\right)^3} + 800 > 0.$$

Thus, $x = \frac{5}{2}$ is a point of minimum.

$$\therefore x^2 = \frac{250}{5} \times 2 \Rightarrow x^2 = 100 \Rightarrow x = 10.$$

OR

Minimum cost is at $h = \frac{5}{2}$.

$$\therefore c(h) = \frac{12500}{h} + 400h^2.$$

$$\therefore c\left(\frac{5}{2}\right) = \frac{12500}{\frac{5}{2}} + 400 \times \frac{25}{4}$$

$$= \frac{12500}{5} \times 2 + 2500 = 5000 + 2500 = ₹\ 7500.$$

Q. 38. A shopkeeper sells three types of flower seeds A_1, A_2, A_3. They are sold in the form of mixture, where the proportions of the seeds are 4 : 4 : 2, respectively. The germination rates of the three types of seeds are 45%, 60% and 35% respectively.

Based on the above information, answer the following:

(*a*) Calculate the probability that a randomly chosen seed will germinate.

(*b*) Calculate the probability that the seed is of type A_2, given that a ramdomly chosen seed germinates.

Ans. Here $P(A_1) = \frac{4}{10}$, $P(A_2) = \frac{4}{10}$ and $P(A_3) = \frac{2}{10}$.

Also $P(G/A_1) = \frac{45}{100}$, $P(G/A_2) = \frac{60}{100}$ and

$P(G/A_3) = \frac{35}{100}$, where G denotes germination.

***(a)*;** $P(G) = P(A_1)\,P(G/A_1) + P(A_2)\,P(G/A_2) + P(A_3)\,P(G/A_3)$

$$= \left(\frac{4}{10}\right)\left(\frac{45}{100}\right)+\left(\frac{4}{10}\right)\left(\frac{60}{100}\right)+\left(\frac{2}{10}\right)\left(\frac{35}{100}\right)$$

$$= \frac{1}{1000}(180+240+70)$$

$$= \frac{490}{1000} = \frac{49}{100} = 0.49.$$

***(b)*;** *By Bayes' Theorem*

$$P(A_2/G) = \frac{P(A_2)\,P(G/A_2)}{P(A_1)P(G/A_1) + P(A_2)\,P(G/A_2) + P(A_3)\,P(G/A_3)}$$

$$= \frac{\left(\frac{4}{10}\right)\left(\frac{60}{100}\right)}{\left(\frac{4}{10}\right)\left(\frac{45}{100}\right)+\left(\frac{4}{10}\right)\left(\frac{60}{100}\right)+\left(\frac{2}{10}\right)\left(\frac{35}{100}\right)}$$

$$= \frac{240}{180+240+70} = \frac{240}{490} = \frac{24}{49}.$$

Holy Faith New Style Sample Paper–11

(Based on the Latest Design & Syllabus Issued by CBSE)

CLASS—12th

SUBJECT—MATHEMATICS

Time Allowed : 3 Hours **Maximum Marks : 80**

General Instructions : Same as Holy Faith New Style Sample Paper–1.

SECTION—A

(Multiple choice questions, each question carries 1 mark)

1. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 + 1$. Then pre-images of 17 and – 3 respectively, are:

(*a*) ϕ, {4, – 4} (*b*) {3, – 3}, ϕ

(*c*) {4, – 4}, ϕ (*d*) {4, – 4}, {2, – 2}.

Ans. (*c*) {4, – 4}, ϕ.

2. $\tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2)$ is equal to :

(*a*) π (*b*) $-\frac{\pi}{3}$

(*c*) $\frac{\pi}{3}$ (*d*) $\frac{2\pi}{3}$.

Ans. (*b*) $-\frac{\pi}{3}$.

3. Given that matrices A and B are of order $3 \times n$ and $m \times 5$ respectively, then the order of the matrix : C = 5A + 3B is :

(*a*) 3×5 and $m = n$ (*b*) 3×5

(*c*) 3×3 (*d*) 5×5.

Ans. (*b*) 3×5.

4. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to :

(*a*) A (*b*) I + A

(*c*) I – A (*d*) I.

Ans. (*d*) I.

5. Let A be a 3×3 matrix such that | adj. A | = 64. Then | A | is equal to:

(*a*) 8 only (*b*) – 8 only

(*c*) 64 (*d*) 8 or – 8.

Ans. (*d*) 8 or – 8.

6. The function $f(x) = |x| - x$ is :

(*a*) continuous but not differentiable at $x = 0$

(*b*) continuous and differentiable at $x = 0$

(*c*) neither continuous nor differentiable at $x = 0$

(*d*) differentiable but not continuous at $x = 0$.

Ans. (*a*) continuous but not differentiable at $x = 0$.

7. The point(*s*), at which the function given by :

$$f(x) = \begin{cases} \frac{x}{|x|}, & x < 0 \\ -1, & x \geq 0 \end{cases} \text{ is :}$$

(*a*) $x \in \mathbf{R}$ (*b*) $x = 0$

(*c*) $x \in \mathbf{R} - \{0\}$ (*d*) $x = -1$ and 1.

Ans. (*a*) $x \in \mathbf{R}$.

8. Rate of change of circumference of a circle with respect to its radius is :

(*a*) π (*b*) 2π

(*c*) $2\pi r$ (*d*) 3π.

Ans. (*b*) 2π.

9. $\int_{-1}^{1} \frac{|x+2|}{x-2} dx$, $x \neq 2$ is equal to:

(*a*) 1 (*b*) –1

(*c*) 2 (*d*) –2.

Ans. (*b*) –1.

10. $\int x^2 e^x \, dx$ is equal to :

(*a*) $e^x(x^2 - 2x - 2) + c$

(*b*) $e^x(x^2 + 2x - 2) + c$

(*c*) $e^x(x^2 + 2x + 2) + c$

(*d*) $e^x(x^2 - 2x + 2) + c$.

Ans. (*d*) $e^x(x^2 - 2x + 2) + c$.

11. The area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, is :

(*a*) 3π (*b*) 2π

(*c*) 6π (*d*) π.

Ans. (*c*) 6π.

12. A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution :

(*a*) $y = vx$ (*b*) $v = yx$

(*c*) $x = vy$ (*d*) $x = v$.

Ans. (*c*) $x = vy$.

13. The degree of differential equation :

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0 \text{ is :}$$

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) Not defined.

Ans. (*d*) Not defined.

14. If $\{\hat{i}, \hat{j}, \hat{k}\}$ are the usual three perpendicular unit vectors, then the value of :

$\hat{i} . (\hat{j} \times \hat{k}) + \hat{j} . (\hat{i} \times \hat{k}) + \hat{k} . (\hat{i} \times \hat{j})$ is :

(*a*) 0 (*b*) – 1

(*c*) 1 (*d*) 3.

Ans. (*d*) 3.

15. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$. If $\vec{b}$ is a vector such that

$\vec{a} . \vec{b} = |\vec{b}|^2$, then $|\vec{b}|$ equals :

(*a*) 7 (*b*) 14

(*c*) $\sqrt{7}$ (*d*) 21.

Ans. (*c*) $\sqrt{7}$.

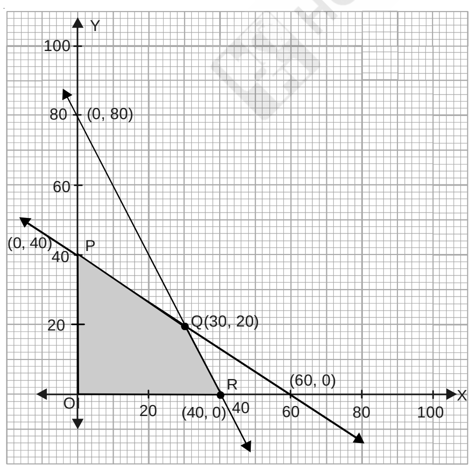
16. The reflection of the point (α, β, γ) in the *xy*-plane is :

(*a*) $(\alpha, \beta, 0)$ (*b*) $(0, 0, \gamma)$

(*c*) $(-\alpha, -\beta, \gamma)$ (*d*) $(\alpha, \beta, -\gamma)$.

Ans. (*d*) $(\alpha, \beta, -\gamma)$.

17. For an L.P.P. the objective function is $Z = 4x + 3y$, and the feasible region determined by a set of constraints (linear inequations) is shown in the graph :



Which one of the following statements is true ?

(*a*) Maximum value of Z is at R

(*b*) Maximum value of Z is at Q

(*c*) Value of Z at R is less than the value at P

(*d*) Value of Z at Q is less than the value at R.

Ans. (*b*) Maximum value of Z is at Q.

18. If P (A) = $\frac{1}{2}$, P (B) = $\frac{3}{8}$ and P (A ∩ B) = $\frac{1}{5}$, then P (B/A) is equal to :

(*a*) $\frac{2}{5}$ (*b*) $\frac{8}{15}$

(*c*) $\frac{2}{3}$ (*d*) $\frac{5}{8}$.

Ans. (*a*) $\frac{2}{5}$.

In the following questions a statements of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer into of the following choices.

(*a*) Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.

(*b*) Both 'A' and 'R' are true but 'R' is not the correct explanation of 'A'.

(*c*) 'A' is true but 'R' is false.

(*d*) 'A' is false but 'R' is true.

19. **Assertion (A) :** Let $f : [0, \infty) \to [0, \infty)$, be a function defined by :

$y = f(x) = x^2$, then $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = 1$

Reason (R) : $\left(\frac{dy}{dx}\right)\left(\frac{dx}{dy}\right) = 1$.

Ans. (*d*) 'A' is false but 'R' is true.

20. **Assertion (A) :** When the shaded region is bounded, then we can find the maximum value of the objective function.

Reason (R) : The Corner Point Method.

Ans. (*a*) Both 'A' and 'R' are true and 'R' is the correct explanation of 'A'.

SECTION—B

(This section comprises of very short answer type questions (VSA) of 2 marks each)

Q. 21. Show that :

$$\tan^{-1}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x; -\frac{1}{\sqrt{2}} \le x \le 1.$$

Ans. LHS = $\tan^{-1}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$$= \tan^{-1}\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}$$

[*Putting* $x = \cos 2\theta$]

$$= \tan^{-1}\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}$$

$$= \tan^{-1}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right) = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

[*Dividing Numerator and Denominator by cos θ*]

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \theta\right)\right]$$

$$= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x = \textbf{RHS.}$$

$$\left[\because \cos 2\theta = x \Rightarrow \theta = \frac{1}{2}\cos^{-1} x\right]$$

Or

Prove the following :

$$\mathbf{\cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1} x\right)\right\}\right] = \sqrt{\frac{1+x^2}{2+x^2}}}.$$

Ans. Put $\cot^{-1}x = \theta$ so that $x = \cot\theta$.

$$\therefore \quad \sin\theta = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \sin(\cot^{-1} x) = \frac{1}{\sqrt{1+x^2}} \quad \text{....(1)}$$

$$\therefore \tan^{-1}\{\sin(\cot^{-1} x)\} = \tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

[*Using (1)*]

$$\text{Put} \quad \tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \phi \quad \text{...(2)}$$

so that $\dfrac{1}{\sqrt{1+x^2}} = \tan\phi$.

$$\therefore \quad \cos\phi = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$$

$$\Rightarrow \cos\left(\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}. \quad [\textit{Using (2)}]$$

$$\text{Hence,} \quad \cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1} x\right)\right\}\right] = \sqrt{\frac{1+x^2}{2+x^2}}.$$

[*Using (1)*]

Q. 22. Prove that $\mathbf{y = \dfrac{4\sin\theta}{(2+\cos\theta)} - \theta}$ **is an increasing function of** $\boldsymbol{\theta}$ **in** $\mathbf{\left[0, \dfrac{\pi}{2}\right]}$**.**

Ans. We have : $f(\theta) = \dfrac{4\sin\theta}{(2+\cos\theta)} - \theta$.

$$\therefore f'(\theta) = 4\frac{(2+\cos\theta)(\cos\theta) - \sin\theta(0-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= 4\frac{2\cos\theta + \cos^2\theta + \sin^2\theta}{(2+\cos\theta)^2} - 1 = 4\frac{2\cos\theta + 1}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} > 0 \text{ for all } \theta \in \left[0, \frac{\pi}{2}\right].$$

Hence, '*f*' is an increasing function of θ in $\left[0, \dfrac{\pi}{2}\right]$.

Q. 23. Find the maximum and minimum values, if any, of the functions given by :

$\boldsymbol{f(x) = x^2, x \in \mathbf{R}}$

Ans. The given function is :

$$f(x) = x^2, x \in \mathbf{R} \quad \text{...(1)}$$

Its graph is as shown in the adjoining figure :

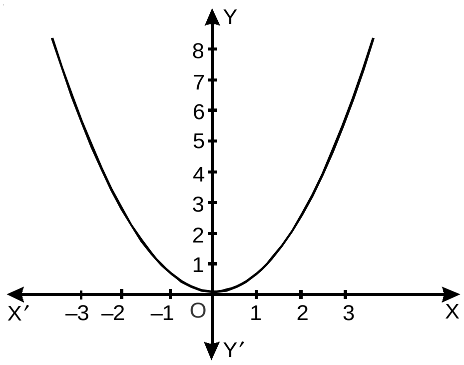


Fig.

From the graph, we have : $f(x) = 0$ when $x = 0$.

Also, $f(x) \geq 0 \ \forall x \in \mathbf{R}$.

Thus minimum value of '*f*' is 0 and point of minimum value of '*f*' is $x = 0$.

And '*f*' has no maximum value and hence no point of maximum value of '*f*' in **R**.

Or

Find the maximum and minimum values, if any, of the functions given by :

$\boldsymbol{f(x) = |x|, x \in \mathbf{R}}$**.**

Ans. The given function is $f(x) = |x|, \forall x \in \mathbf{R}$ ...(1)

Its graph is as shown in the following figure :

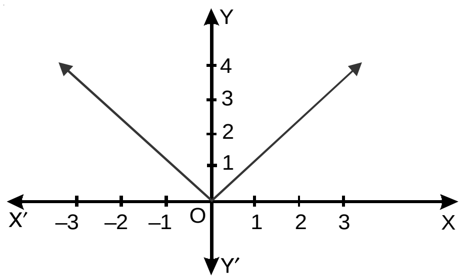


Fig.

From the graph, we have :

$f(x) \geq 0 \ \forall\ x \in \mathbf{R}$ and $f(x) = 0$ when $x = 0$.

Thus minimum value of 'f' is 0 and point of minimum value of 'f' is $x = 0$.

And 'f' has no maximum value and has no point of maximum value of 'f' in **R.**

Q. 24. Evaluate : $\displaystyle\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$.

Ans. Let $I = \displaystyle\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$.

Put $\sqrt{\dfrac{x}{a+x}} = \sin\theta$ *i.e.* $\dfrac{x}{a+x} = \sin^2\theta$

$\Rightarrow \quad x = a\sin^2\theta + x\sin^2\theta$

$\Rightarrow x(1-\sin^2\theta) = a\sin^2\theta \Rightarrow x\cos^2\theta = a\sin^2\theta$

$\Rightarrow \quad x = a\tan^2\theta$

so that $\quad dx = 2a\tan\theta\sec^2\theta\,d\theta$.

When $x = 0$, $\sin\theta = 0 \quad \Rightarrow \theta = 0$.

When $x = a$, $\sin\theta = \sqrt{\dfrac{a}{2a}} = \dfrac{1}{\sqrt{2}} \quad \Rightarrow \theta = \dfrac{\pi}{4}$.

$$\therefore \quad I = \int_0^{\pi/4} \theta(2a\tan\theta\sec^2\theta)\,d\theta$$

$$= 2a\int_0^{\pi/4} \theta(\tan\theta\sec^2\theta)\,d\theta$$

$$= 2a\left[\left\{\theta\frac{\tan^2\theta}{2}\right\}_0^{\pi/4} - \int_0^{\pi/4}(1)\frac{\tan^2\theta}{2}\,d\theta\right]$$

[*Integrating by parts*]

$$= 2a\left[\left(\frac{\pi}{8}(1)-0\right) - \frac{1}{2}\int_0^{\pi/4}(\sec^2\theta - 1)\,d\theta\right]$$

$$= a\left[\frac{\pi}{4} - \int_0^{\pi/4}(\sec^2\theta-1)\,d\theta\right]$$

$$= a\left[\frac{\pi}{4} - \{\tan\theta - \theta\}_0^{\pi/4}\right]$$

$$= a\left[\frac{\pi}{4} - \left\{\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - (0-0)\right\}\right]$$

$$= a\left[\frac{\pi}{4} - \left(1 - \frac{\pi}{4}\right)\right]$$

$$= a\left(\frac{\pi}{2} - 1\right) = \frac{a}{2}(\pi - 2).$$

Q. 25. If $y = 500\,e^{7x} + 600\,e^{-7x}$, show that $\dfrac{d^2y}{dx^2} = 49y$.

Ans. We have : $y = 500\,e^{7x} + 600\,e^{-7x}$ ...(1)

$\therefore \quad \dfrac{dy}{dx} = 500.e^{7x}.7 + 600\,e^{-7x}.(-7)$

$= 7(500\,e^{7x} - 600\,e^{-7x})$ and

$\dfrac{d^2y}{dx^2} = 7(500\,e^{7x}.7 - 600\,e^{-7x}(-7))$

$= 49(500\,e^{7x} + 600\,e^{-7x})$

$= 49y.$ [*Using (1)*]

Hence, $\dfrac{d^2y}{dx^2} = 49y$.

SECTION—C

(This section comprises of short answer type questions (SAQ) of 3 marks each)

Q. 26. Integrate the rational function $\dfrac{1}{x(x^4-1)}$.

Ans. $\quad I = \displaystyle\int \frac{1}{x(x^4-1)}\,dx$

$$= \int \frac{x^3}{x^4(x^4-1)}\,dx.$$

[*Multiplying Num. and Denom. by x^3*]

Put $x^4 = t$ so that $4x^3\,dx = dt$

$\Rightarrow x^3\,dx = \dfrac{1}{4}dt$.

$$\therefore \quad I = \int \frac{\frac{1}{4}dt}{t(t-1)}$$

$$= -\frac{1}{4}\int\left(\frac{1}{t} - \frac{1}{t-1}\right)dt \quad \text{[Partial Fractions]}$$

$$= -\frac{1}{4}\int\frac{1}{t}\,dt + \frac{1}{4}\int\frac{1}{t-1}\,dt$$

$$= -\frac{1}{4}\log|t| + \frac{1}{4}\log|t-1| + c$$

$$= \frac{1}{4}\log\left|\frac{t-1}{t}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + c.$$

Q. 27. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly ?

Ans. Let the events be :

E_1 : ''the student knows the answer''
E_2 : ''the student guesses the answer'' and
A : ''the student answers correctly''.

$$\therefore \quad P(E_1) = \frac{3}{4}, \; P(E_2) = \frac{1}{4}.$$

$$P(A/E_2) = \frac{1}{4} \text{ and}$$

$$P(A/E_1) = 1.$$

By *Bayes' Theorem,*

$P(E_1/A)$

$$= \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)}$$

$$= \frac{\left(\frac{3}{4}\right)(1)}{\left(\frac{3}{4}\right)(1) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)} = \frac{3}{\frac{13}{4}} = \frac{12}{13}.$$

Q. 28. Evaluate : $\int \frac{\cos x + 2\sin x + 3}{4\cos x + 5\sin x + 6}\,dx.$

Ans. Let $I = \int \frac{\cos x + 2\sin x + 3}{4\cos x + 5\sin x + 6}\,dx.$

Let A, B and C be constants, such that :

$$\cos x + 2\sin x + 3 = A(4\cos x + 5\sin x + 6) + B\frac{d}{dx}(4\cos x + 5\sin x + 6) + C$$

$= 4A\cos x + 5A\sin x + 6A + B(-4\sin x + 5\cos x) + C$

$= (4A + 5B)\cos x + (5A - 4B)\sin x + 6A + C.$

Comparing, $4A + 5B = 1$ ...(1)

$5A - 4B = 2$ ...(2)

and $6A + C = 3$ ...(3)

$(1) \times 5 \Rightarrow 20A + 25B = 5$ ...(4)

$(2) \times 4 \Rightarrow 20A - 16B = 8$ ...(5)

$(4) - (5) \Rightarrow 41B = -3$

$$\Rightarrow \quad B = -\frac{3}{41}.$$

Putting in (1), $4A + 5\left(-\frac{3}{41}\right) = 1$

$$\Rightarrow \quad A = \frac{1 + \left(\frac{15}{41}\right)}{4} = \frac{14}{41}.$$

Putting in (3), $6\left(\frac{14}{41}\right) + C = 3$

$$\Rightarrow C = 3 - \left(\frac{84}{41}\right) = \frac{39}{41}.$$

$$\therefore I = \int \frac{A(4\cos x + 5\sin x + 6)}{(4\cos x + 5\sin x + 6)}\,dx + \int \frac{B(-4\sin x + 5\cos x)}{4\cos x + 5\sin x + 6}\,dx + \int \frac{C}{4\cos x + 5\sin x + 6}\,dx$$

$$= Ax + B\log|4\cos x + 5\sin x + 6| + C\int \frac{dx}{4\cos x + 5\sin x + 6} \quad ...(6)$$

Let $I_1 = \int \frac{dx}{4\cos x + 5\sin x + 6}$

$$= \int \frac{dx}{4\left(\frac{1 - \tan^2 \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x}\right) + 5\left(\frac{2\tan \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x}\right) + 6}$$

$$= \int \frac{\left(1 + \tan^2 \frac{1}{2}x\right)dx}{4 - 4\tan^2 \frac{1}{2}x + 10\tan \frac{1}{2}x + 6 + 6\tan^2 \frac{1}{2}x}$$

$$= \int \frac{\sec^2 \frac{1}{2}x}{2\tan^2 \frac{1}{2}x + 10\tan \frac{1}{2}x + 10}\,dx.$$

Put $\tan \frac{1}{2}x = z$

so that $\frac{1}{2}\sec^2 \frac{1}{2}x\,dx = dz$ *i.e.* $\sec^2 \frac{1}{2}x\,dx = 2dz.$

$$\therefore I_1 = \int \frac{2dz}{2z^2 + 10z + 10} = \int \frac{dz}{z^2 + 5z + 5}$$

$$= \int \frac{dz}{\left(z^2 + 5z + \frac{25}{4}\right) + 5 - \frac{25}{4}}$$

$$= \int \frac{dz}{\left(z + \frac{5}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2}$$

Put $z + \frac{5}{2} = u$ so that $dz = du.$

$$\therefore I_1 = \int \frac{du}{u^2 - \left(\frac{\sqrt{5}}{2}\right)^2}$$

''**Form :** $\int \frac{1}{x^2 - a^2}\,dx$''

$$= \frac{1}{2\left(\frac{\sqrt{5}}{2}\right)} \log \left| \frac{u - \sqrt{5}/2}{u + \sqrt{5}/2} \right| = \frac{1}{\sqrt{5}} \log \left| \frac{2u - \sqrt{5}}{2u + \sqrt{5}} \right|$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{2z + 5 - \sqrt{5}}{2z + 5 + \sqrt{5}} \right|$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{2 \tan \frac{x}{2} + 5 - \sqrt{5}}{2 \tan \frac{x}{2} + 5 + \sqrt{5}} \right|$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{2 \sin \frac{1}{2} x + (5 - \sqrt{5}) \cos \frac{1}{2} x}{2 \sin \frac{1}{2} x + (5 + \sqrt{5}) \cos \frac{1}{2} x} \right|.$$

Putting the values of A, B and C and I_1 in (6), we get :

$$I = \frac{14}{41} x - \frac{3}{41} \log |4 \cos x + 5 \sin x + 6|$$

$$+ \frac{39}{41\sqrt{5}} \log \left| \frac{2 \sin \frac{1}{2} x + (5 - \sqrt{5}) \cos \frac{1}{2} x}{2 \sin \frac{1}{2} x + (5 + \sqrt{5}) \cos \frac{1}{2} x} \right| + c.$$

Or

By using properties of definite integral, evaluate the integral :

$$\int_0^{\pi} \log (1 + \cos x)\, dx.$$

Ans. $I = \int_0^{\pi} \log (1 + \cos x)\, dx$ ...(1)

$\therefore$ $I = \int_0^{\pi} \log [1 + \cos (\pi - x)]\, dx$ [*Property V*]

$\Rightarrow$ $I = \int_0^{\pi} \log (1 - \cos x)\, dx$ ...(2)

Adding (1) and (2), we get :

$$2I = \int_0^{\pi} [\log (1 + \cos x)(1 - \cos x)]\, dx$$

$$= \int_0^{\pi} \log (1 - \cos^2 x)\, dx$$

$$= \int_0^{\pi} \log \sin^2 x\, dx$$

$$= 2\int_0^{\pi} \log \sin x\, dx$$

$\Rightarrow$ $I = \int_0^{\pi} \log \sin x\, dx$

$= 2\int_0^{\pi/2} \log \sin x\, dx$ ...(3)

[$\because$ *log (sin (π – x) = log sin x*]

$= 2\int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$

[*Property V*]

$= 2\int_0^{\pi/2} \log \cos x\, dx$ ...(4)

Adding (3) and (4),

$$2I = 2\int_0^{\pi/2} \log \sin x \cos x\, dx$$

$\Rightarrow$ $I = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$

$= \int_0^{\pi/2} \log \sin 2x\, dx - \log 2 \int_0^{\pi/2} 1.dx$

$= I_1 - \log 2\, [x]_0^{\pi/2} = I_1 - \frac{\pi}{2} \log 2$ ....(5)

Now $I_1 = \int_0^{\pi/2} \log \sin 2x\, dx.$

Put 2x = t so that $2\, dx = dt$ *i.e.* $dx = \frac{1}{2} dt$.

When $x = 0$, $t = 0$, when $x = \frac{\pi}{2}$, $t = \pi$.

$\therefore$ $I_1 = \frac{1}{2}\int_0^{\pi} \log \sin t\, dt = \frac{1}{2}\int_0^{\pi} \log \sin x\, dx$

$= \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin x\, dx$

$\Rightarrow$ $I_1 = \frac{1}{2} I.$

From (5), $I = \frac{1}{2} I - \frac{\pi}{2} \log 2$

$\Rightarrow$ $\frac{1}{2} I = -\frac{\pi}{2} \log 2.$

Hence, $I = -\pi \log 2.$

Q. 29. Show that the differential equation $x^2 \frac{dy}{dx} = x^2 - 2y^2 + xy$ is homogeneous.

Ans. The given equation is $x^2 \frac{dy}{dx} = x^2 - 2y^2 + xy$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2} \quad ...(1)$$

Here $f(x, y) = \frac{x^2 - 2y^2 + xy}{x^2}$.

$$\therefore \quad f(\lambda x, \lambda y) = \frac{\lambda^2 x^2 - 2\lambda^2 y^2 + (\lambda x)(\lambda y)}{\lambda^2 x^2}$$

$$= \frac{\lambda^2 \left(x^2 - 2y^2 + xy\right)}{\lambda^2 x^2}$$

$$= \lambda^0 \frac{x^2 - 2y^2 + xy}{x^2}$$

$$= \lambda^0 f(x, y).$$

Thus $f(x, y)$ is homogeneous function of degree zero.

To solve :

Put y = vx so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore$ (1) becomes :

$$v + x\frac{dv}{dx} = \frac{x^2 - 2v^2x^2 + x(vx)}{x^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = 1 - 2v^2 + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = 1 - 2v^2$$

$$\Rightarrow \quad \frac{dv}{1 - 2v^2} = \frac{dx}{x}.$$ | ***Variables Separable***

Integrating, $\int \frac{dv}{1 - 2v^2} = \int \frac{dx}{x} + c$

$$\Rightarrow \frac{1}{2}\int \frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2 - v^2} = \int \frac{dx}{x} + c$$

$$\Rightarrow \frac{1}{2} \cdot \frac{1}{2\frac{1}{\sqrt{2}}} \log\left|\frac{\frac{1}{\sqrt{2}} + v}{\frac{1}{\sqrt{2}} - v}\right| = \log|x| + c$$

$$\Rightarrow \frac{1}{2\sqrt{2}} \log\left|\frac{1 + \sqrt{2}\,v}{1 - \sqrt{2}\,v}\right| = \log|x| + c$$

$$\Rightarrow \frac{1}{2\sqrt{2}} \log\left|\frac{1 + \sqrt{2}\,\frac{y}{x}}{1 - \sqrt{2}\,\frac{y}{x}}\right| = \log|x| + c$$

$$\Rightarrow \frac{1}{2\sqrt{2}} \log\left|\frac{x + \sqrt{2}\,y}{x - \sqrt{2}\,y}\right| = \log|x| + c,$$

which is the reqd. solution.

Or

Show that the differential equation $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$ **is homogeneous.**

Ans. The given equation is :

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{y}}} = \frac{\left(\frac{x}{y} - 1\right)e^{\frac{x}{y}}}{1 + e^{\frac{x}{y}}} \quad ...(1)$$

Here $f(x, y) = \frac{\left(\frac{x}{y} - 1\right)e^{\frac{x}{y}}}{1 + e^{\frac{x}{y}}}$.

$$\therefore \quad f(\lambda x, \lambda y) = \frac{\left(\frac{\lambda x}{\lambda y} - 1\right)e^{\frac{\lambda x}{\lambda y}}}{1 + e^{\frac{\lambda x}{\lambda y}}}$$

$$= \frac{\left(\frac{x}{y} - 1\right)e^{\frac{x}{y}}}{1 + e^{\frac{x}{y}}} = \lambda^0 f(x, y).$$

Thus $f(x, y)$ is homogeneous function of degree zero.

To solve :

Put x = vy so that $\frac{dx}{dy} = v + y\frac{dv}{dy}$.

$\therefore$ (1) becomes.

$$v + y\frac{dv}{dy} = \frac{\left(\frac{vy}{y} - 1\right)e^{\frac{vy}{y}}}{1 + e^{\frac{vy}{y}}}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{(v - 1)e^v}{1 + e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{(v - 1)e^v}{1 + e^v} - v$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{v e^v - e^v - v - v e^v}{1 + e^v}$$

$$= -\frac{e^v + v}{1 + e^v}$$

$$\Rightarrow \frac{1+e^v}{v+e^v}\,dv = -\frac{dy}{y} \qquad \text{| Variables Separable}$$

Integrating,

$$\int \frac{1+e^v}{v+e^v}\,dv = -\int \frac{1}{y}\,dy + \log|c|.$$

Put v + e^v = t so that $(1 + e^v)\, dv = dt$.

$$\therefore \int \frac{dt}{t} = -\int \frac{1}{y}\,dy + \log|c|$$

$$\Rightarrow \log|t| = -\log|y| + \log|c|$$

$$\Rightarrow \log|t| + \log|y| = \log|c|$$

$$\Rightarrow \log|ty| = \log|c|$$

$$\Rightarrow ty = c \Rightarrow (v + e^v)\, y = c$$

$$\Rightarrow \left(\frac{x}{y} + e^{\frac{x}{y}}\right) y = c \Rightarrow x + y\, e^{\frac{x}{y}} = c,$$

which is the reqd. solution.

Q. 30. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that y = 0 when x = 0.

Ans. The given equation is :

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y$$

$$\Rightarrow \frac{dy}{dx} = e^{3x+4y}$$

$$\Rightarrow e^{-4y}\, dy = e^{3x}\, dx. \qquad \text{| Variables Separable}$$

Integrating, $\int e^{-4y}\, dy = \int e^{3x}\, dx + c$

$$\Rightarrow \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + c \qquad ...(1)$$

When $x = 0$, $y = 0$,

$$\therefore -\frac{1}{4} = \frac{1}{3} + c$$

$$\Rightarrow c = -\frac{1}{4} - \frac{1}{3} = -\frac{7}{12}.$$

Putting in (1), $-\frac{1}{4}\cdot\frac{1}{e^{4y}} = \frac{1}{3}e^{3x} - \frac{7}{12}$

$\Rightarrow -3\, e^{-4y} = 4\, e^{3x} - 7$

$\Rightarrow 4\, e^{3x} + 3\, e^{-4y} = 7,$

which is the reqd. particular solution.

Or

If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Hence, solve the system of equations :

$x - 2y = 10$
$2x - y - z = 8$
$-2y + z = 7$.

Ans. Here, $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$.

$$\therefore |A| = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= 1(-1-2) - 2(-2+0)$$
$$= -3 + 4 = 1 \neq 0.$$

$\therefore$ A is non-singular $\Rightarrow A^{-1}$ exists.

Here $A_{11} = -3,\ A_{12} = 2,\ A_{13} = 2$;
$A_{21} = -2,\ A_{22} = 1,\ A_{23} = 1$;
$A_{31} = -4,\ A_{32} = 2,\ A_{33} = 3$.

$$\therefore \text{adj. A} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

$$\therefore A^{-1} = \frac{1}{|A|}(\text{Adj}\, A)$$

$$= \frac{1}{1}\begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad ...(1)$$

The given system of equations can be written as :

$$A'X = B,$$

where $A' = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$.

$$\therefore X = (A')^{-1} B = (A^{-1})' B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -30+16+14 \\ -20+8+7 \\ -40+16+21 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}.$$

Comparing, $x = 0$, $y = -5$ and $z = -3$.

Q. 31. Find the maximum and minimum values of $f : x + 2y$ subject to the constraints :

$x + 2y \geq 100,\ 2x - y \leq 0,\ 2x + y \leq 200,\ x, y \geq 0$.

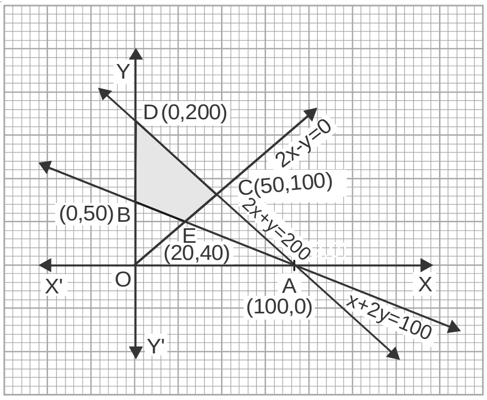
Ans. The system of constraints is :

$x + 2y \geq 100$ ...(1)

$2x - y \leq 0$ ...(2)

$2x + y \leq 200$ ...(3)

and $x, y \geq 0$ ...(4)



The shaded region in the above figure is the feasible region determined by the system of constraints (1)–(4).

It is observed that the feasible region ECDB is bounded. Thus we use ***Corner Point Method*** to determine the maximum of Z, where :

$$Z = x + 2y \qquad ...(5)$$

The co-ordinates of E, C, D and B are

(20, 40)(on solving $x + 2y = 100$ and $2x - y = 0$)

(50, 100)(on solving $2x + y = 200$ and $2x - y = 0$)

(0, 200) and (0, 50) respectively.

Corner point	Corresponding Value of Z
E : (20, 40)	**100 (Minimum)**
C : (50, 100)	250
D : (0, 200)	**400 (Maximum)**
B : (0, 50)	**100 (Minimum)**

Hence, $Z_{max} = 400$ at (0, 200)

and $Z_{min} = 100$ at all points on the line segment joining the points B (0, 50) and E (20, 40).

SECTION—D

(This section comprises of Long Answer Type questions (LAQ) of 5 marks each)

Q. 32. Find the area of the region bounded by the curve $ay^2 = x^3$, the *y*-axis and the lines $y = a$ and $y = 2a$.

Ans. Reqd. area $= \int_a^{2a} (ay^2)^{1/3} dy$

$$= a^{1/3} \int_a^{2a} y^{2/3} dy$$

$$= a^{1/3} \left[\frac{y^{5/3}}{5/3}\right]_a^{2a}$$

$$= \frac{3}{5} a^{1/3} \left[(2a)^{5/3} - (a^{5/3})\right]$$

$$= \frac{3}{5} a^{1/3} a^{5/3} \left[2^{5/3} - 1\right]$$

$$= \frac{3}{5} a^{2} \left[2^{5/3} - 1\right] \text{ sq. units.}$$

Q. 33. Let A = {$x \in Z : 0 \le x \le 12$}. Show that R = {$(a, b) : a, b \in A : |a - b|$ is divisible by 4} is an equivalence relation. Find the set of all elements related to 1. Also, write the equivalence class [2].

Ans. We have :

R = {$(a. b) : a, b \in A : |a - b|$ is divisible by 4}.

(*i*) Reflexive : For any $a \in A$,

$\therefore \quad (a, a) \in R.$

$| a - a | = 0$, which is divisible by 4.

Thus, *R is reflexive*.

Symmetric : Let $(a, b) \in R$

$\Rightarrow | a - b |$ is divisible by 4

$\Rightarrow | b - a |$ is divisible by 4.

Thus, *R is symmetric*.

Transitive : Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow | a - b |$ is divisible by 4 and $| b - c |$ is divisible by 4

$\Rightarrow \quad | a - b | = 4\lambda \qquad ...(1)$

$\Rightarrow \quad a - b = \pm 4\lambda$

and $\quad | b - c | = 4\mu$

$\Rightarrow \quad b - c = \pm 4\mu \qquad ...(2)$

Adding (1) and (2),

$(a - b) + (b - c) = \pm 4(\lambda + \mu)$

$\Rightarrow \quad a - c = \pm 4(\lambda + \mu)$

$\Rightarrow \quad (a - c) \in R.$

Thus, *R is transitive*.

Now, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

(*ii*) Let 'x' be an element of A such that $(x, 1) \in R$

$\Rightarrow | x - 1|$ is divisible by 4

$\Rightarrow \quad x - 1 = 0, 4, 8, 12$

$\Rightarrow \quad x = 1, 5, 9.$

Hence, the set of all elements of A which are related to 1 is {1, 5, 9}.

(*iii*) Let $(x, 2) \in R$.

Then, $\quad | x - 2 | = 4k,$

where $\quad k \le 3.$

$\therefore \quad x = 2, 6, 10.$

Hence, equivalence class [2] = [2, 6, 10].

Or

The Earth has 24 time zones, defined by dividing the Earth into 24 equal longitudinal segments. These are regions on Earth that have the same standard time.

For example. USA and India fall in different time zones, but Sri Lanka and India are in the same time zone.

A relation R is defined as the set

U = {All people on the earth} such that R = {(x, y)} the time difference between the time zones x and y reside in is 6 hours}.

(*i*) Check whether the relation R is reflexive, symmetric and transitive.

(*ii*) Is relation R an equivalence relation ?

Ans. (*i*) For no. $x \in U, (x, x) \in R$.

[∵ *Difference in time between x and x is 0 hours*]

Thus, R is *not reflexive*.

(*ii*) Now $(x, y) \in R \Rightarrow (y, x) \in R$.

[∵ *When the difference in time between x and y is 6 hours, then the difference in time between y and x is 6 hours*]

Thus, R is *symmetric.*

(*iii*) Now $(x, y) \in R$ and $(y, z) \in R$ but $(x, z) \notin R$.

[∵ *If the difference in time between x and y is 6 hours and the difference in time between y and z is also 6 hours, then the difference in time between x and z is ether 0 hours or 12 hours*]

Thus, R is *not transitive.*

(*ii*) From part (*i*), it is concluded that *R is not an equivalence relation.*

Q. 34. Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

Ans. Here $\vec{AB} = (2\hat{i} + 3\hat{j} + 5\hat{k}) - (\hat{i} + \hat{j} + 2\hat{k}) = \hat{i} + 2\hat{j} + 3\hat{k}$

and $\vec{AC} = (1\hat{i} + 5\hat{j} + 5\hat{k}) - (\hat{i} + \hat{j} + 2\hat{k}) = 4\hat{j} + 3\hat{k}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

But $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$

$= \hat{i}(6 - 12) - \hat{j}(3 - 0) + \hat{k}(4 - 0)$

$= -6\hat{i} - 3\hat{j} + 4\hat{k}$.

$\therefore$ Reqd. area of $\Delta ABC = \frac{1}{2} |-6\hat{i} - 3\hat{j} + 4\hat{k}|$

$= \frac{1}{2}\sqrt{36 + 9 + 16}$

$= \frac{1}{2}\sqrt{61}$ sq. units.

Or

Show that the line joining the mid-points of the two sides of the triangle is parallel to the third side and is half of its length.

Ans. Let A (x_1, y_1, z_1), B (x_2, y_2, z_2) and C (x_3, y_3, z_3) be the vertices of ΔABC. If D, E are mid-points of [AB] and [AC] respectively, then their co-ordinates are :

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

and $\left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}, \frac{z_1 + z_3}{2}\right)$ respectively.

Now direction-ratios of BC are :

$< x_3 - x_2, y_3 - y_2, z_3 - z_2 >$

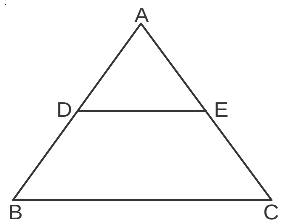


Fig.

and direction-ratios of DE are :

$$< \frac{x_1 + x_3}{2} - \frac{x_1 + x_2}{2}, \frac{y_1 + y_3}{2} - \frac{y_1 + y_2}{2}, \frac{z_1 + z_3}{2} - \frac{z_1 + z_2}{2} >$$

i.e. $< \frac{x_3 - x_2}{2}, \frac{y_3 - y_2}{2}, \frac{z_3 - z_2}{2} >$

i.e. $< x_3 - x_2, y_3 - y_2, z_3 - z_2 >$.

Thus DE || BC.

Now

$$|DE| = \sqrt{\left(\frac{x_1 + x_3}{2} - \frac{x_1 + x_2}{2}\right)^2 + \left(\frac{y_1 + y_3}{2} - \frac{y_1 + y_2}{2}\right)^2 + \left(\frac{z_1 + z_3}{2} - \frac{z_1 + z_2}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2} = \frac{1}{2}|BC|.$$

Hence, DE || BC and $|DE| = \frac{1}{2}|BC|$.

Q. 35. (*i*) Find the probability that both children are males, if it is known that atleast one of the children is male.

(*ii*) Find the probability that both children are females, if it is known that the elder child is a female.

Ans. (*i*) Here S = {(M, M), (M, F), (F, M)}

and A = {(M, M)}.

$\therefore$ Reqd. probability $= \frac{n(A)}{n(S)} = \frac{1}{3}$.

(*ii*) Here S = {(M, M), (M, F)}

and A = {(M, M)}.

$\therefore$ Reqd. probability $= \frac{n(A)}{n(S)} = \frac{1}{2}$.

(*b*) Here S = {(M, F), (F, M), (F, F)}

and A = {(F, F)}.

$\therefore$ Required probability $= \frac{n(A)}{n(S)} = \frac{1}{3}$.

SECTION—E

(This section comprises of 4 case-study/passage bases question of 4 marks each into such-parts)

Q. 36. • Derivative of $y = f(x)$, if it exists, is called the derivative of first order and is denoted by $\frac{dy}{dx}$ or $f'(x)$.

• **Derivative of $\frac{dy}{dx}$ or $f'(x)$ is called derivative of second order and is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$.**

Based on above information, answer the following :

(I) If $y = x^2 + 3x + 2$, find $\frac{d^2y}{dx^2}$.

(II) If $y = \tan x + \sec x$, then find $\frac{d^2y}{dx^2}$.

(III) If $f(x) = \log x$, then find $f''(x)$.

Or

If $y = ae^{2x} + be^{-x}$, then find $\frac{d^2y}{dx^2} - \frac{dy}{dx}$.

Ans. (I) Here, $y = x^2 + 3x + 2$

$\therefore \quad \frac{dy}{dx} = 2x + 3$

Hence, $\frac{d^2y}{dx^2} = 2.$

(II) Here, $y = \tan x + \sec x$

$\therefore \quad \frac{dy}{dx} = \sec^2 x + \sec x \tan x$

and $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$.

(III) Here, $f(x) = \log x$

$\therefore \quad f'(x) = \frac{1}{x}$

and $f''(x) = \frac{1}{x^2}$.

Or

Here, $y = ae^{2x} + be^{-x}$

$\therefore \quad \frac{dy}{dx} = 2ae^{2x} - be^{-x}$

and $\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$

$\therefore \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} = 2ae^{2x} + 2be^{-x}$

$= 2(ae^{2x} + be^{-x}) = 2y.$

Q. 37. A thermometer reading 80°F is taken outside. 5 minutes later the thermometer reads 60°F. After another 5 minutes the thermometer reads 50°F. At any time *t*, the thermometer reading be T°F and the outside temperature be S°F.

Based on the above information, answer the following questions.

(I) If λ is positive constant of proportionality, then find $\frac{dT}{dt}$.

(II) Find the value of T(5) is :

(III) Find the general solution of differential equation formed in given situation.

Or

Find the value of constant of integration *c* in the solution of differential equation formed in given situation.

Ans. (I) Given, at any time *t* the thermometer reading is T°F and the outside temperature is S°F. By *Newton's Law of Cooling*, $\frac{dT}{dt} = -\lambda(T - S)$.

(II) Since, after 5 minutes, thermometer reads 60°F,

$\therefore$ value of T(5) = 60°F.

(III) We have : $\frac{dT}{dt} = -\lambda(T - S)$

$\Rightarrow \frac{dT}{T-S} = -\lambda dt \Rightarrow \int \frac{1}{T-S} dT = -\lambda \int dt$

$\Rightarrow \log(T - S) = -\lambda t + c.$

Or

Since, at $t = 0$, T = 80°F.

$\therefore \log(80 - S) = 0 + c \Rightarrow c = \log(80 - S).$

Q. 38. Read the following passage and answer the questions given below :

In an Office three employees James, Sophia and Oliver process incoming copies of a certain form. James processes 50% of the forms, Sophia processes 20% and Oliver the remaining 30% of the forms. James has an error rate of 0.06 , Sophia has an error rate of 0.04 and Oliver has an error rate of 0.03.

Based on the above information, answer the following questions.

(I) Find the probability that Sophia processed the form and committed an error.

(II) Find the total probability of committing an error in processing the form.

(III) The manager of the Company wants to do a quality check. During inspection, he selects a form at random from the days output of processed form. If the form selected at random has an error, find the probability that the form is not processed by James.

Or

Let E be the event of committing an error in processing the form and let E_1, E_2 and E_3 be the events that Jayant, Sonia and Oliver processed the form. Find the value of $\sum_{i=1}^{3} P(E_i | E)$.

Ans. Let the events be as below:

E_1 : James processed the form

E_2 : Sophia processed the form

E_3 : Oliver processed the form

and E : Error was committed.

We have : $P(E_1) = \dfrac{50}{100} = \dfrac{5}{10}$,

$P(E_2) = \dfrac{20}{100} = \dfrac{1}{5}$

and $P(E_3) = \dfrac{30}{100} = \dfrac{3}{10}$.

Also, $P(E/E_1) = 0.06$,

$P(E/E_2) = 0.04$ and $P(E/E_3) = 0.03$

(I) $P(E \cap E_2) = P(E_2) \,.\, P(E/E_2) = \dfrac{1}{5} \times 0.04$

$= 0.008.$

(II) Total probability of committing an error in processing the form is given by

$$P(E) = P(E_1)\, P(E/E_1) + P(E_2)\, P(E/E_2) + P(E_3)\, P(E/E_3)$$

$$= \frac{50}{100} \times 0.06 + \frac{20}{100} \times 0.04 + \frac{30}{100} \times 0.03$$

$$= 0.0047.$$

(III) Probability that the form is processed by James, given that the form has an error is given by :

$P(E_1/E)$

$$= \frac{P(E/E_1) \times P(E_1)}{P(E/E_1)\,.\,P(E_1) + P(E/E_2)\,.\,P(E_2) + P(E/E_3).\;P(E_3)}$$

$$= \frac{0.06 \times \frac{50}{100}}{0.06\,\frac{50}{100} + 0.04 \times \frac{20}{100} - 0.03 \times \frac{30}{100}} = \frac{30}{47}$$

Hence, the required probability that the form is not processed by James, given that the form has an error

$= P(\bar{E_1} \mid E) = 1 - P(E_1 \mid E) = 1 - \dfrac{30}{47} = \dfrac{17}{47}$.